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DOCTORAL THESIS

ON THE OUTER SYNCHRONIZATION OF COMPLEX DYNAMICAL NETWORKS

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Dedicated to my supervisor

Dr. Joaquín Míguez

from whom I learnt so much, not only with respect to academia,

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Abstract

Complex network models have become a major tool in the modeling and analysis of many physical, biological and social phenomena. A complex network exhibits behaviors which emerge as a consequence of interactions between its constituent elements, that is, remarkably, not the same as individual components.

One particular topic that has attracted the researchers' attention is the analysis of how synchronization occurs in this class of models, with the expectation of gaining new insights of the interactions taking place in real-world complex systems. Most of the work in the literature so far has been focused on the synchronization of a collection of interconnected nodes (forming one single network), where each node is a dynamical system governed by a set of nonlinear differential equations, possibly displaying chaotic dynamics.

In this thesis, we study an extended version of this problem. In particular, we consider a setup consisting of two complex networks which are coupled unidirectionally, in such a way that a set of signals from the *master* network are injected into the *response* network, and then investigate how synchronization is attained. Our analysis is fairly general. We impose few conditions on the network structure and do not assume that the nodes in *a single network* are synchronized.

This work can be divided into two main parts; outer synchronization in fractional-order networks, and outer synchronization in ordinary networks. In both cases the system parameters are perturbed by bounded, time varying and unknown perturbations. The synchronizer feedback matrix is possibly perturbed with the same type of perturbations as well. In both cases, of fractional-order and ordinary networks, we build up several theorems that ensure the attainment of synchronization in various scenarios, including, e.g., cases in which the coupling matrix of the networks is non-diffusive (hence we can avoid this assumption, which is almost invariably made in the literature). In all the cases of interest, we show that the scheme for coupling the networks is very simple, as it reduces to the computation of a single gain matrix whose dimension is independent of the number of network nodes. The structure of the designed synchronizer is also very simple, making it convenient for real-world applications.

Although all of the proposed schemes are assessed analytically, numerical results (obtained by computer simulations) are also provided to illustrate the

proposed methods.

Resumen

Las redes complejas se han convertido en una herramienta fundamental en el anlisis de muchos sistemas físicos, biológicos y sociales. Una red compleja presenta comportamientos que “emergen” como consecuencia de las interacciones entre sus elementos constituyentes pero que no se observan de forma individual en estos elementos.

Un aspecto en concreto que ha atrapado la atención de muchos investigadores es el análisis de cómo se producen fenómenos de sincronización en esta clase de modelos, con la esperanza de alcanzar una mayor comprensión de las interacciones que tienen lugar en sistemas complejos del mundo real. La mayor parte del trabajo publicado hasta ahora ha estado centrado en la sincronización de una colección de nodos interconectados (que forman una nica red con entidad propia), donde cada nodo es un sistema dinmico gobernado por un conjunto de ecuaciones diferenciales no lineales, posiblemente caóticas.

En esta tesis estudiamos una versión extendida de este problema. En concreto, consideramos un sistema formado por dos redes complejas acopladas unidireccionalmente, de manera que un conjunto de señales de la red principal se inyectan en la red secundaria, e investigamos cómo se alcanza un estado de sincronización. Este fenómeno se conoce como “sincronización extern”. Nuestro anlisis es muy general. Se imponen pocas condiciones a las estructura de las redes y no es necesario suponer que los nodos de cada red estén sincronizados entre sí previamente.

Esta memoria se puede dividir en dos bloques: la sincronización externa de redes descritas por ecuaciones diferenciales de orden fraccionario y la sincronización externa de redes ordinarias (descritas por ecuaciones diferenciales de orden entero). En ambos casos, se admite que los parmetros del sistema puedan estar sujetos a perturbaciones desconocidas, posiblemente variables con el tiempo, pero acotadas. La matriz de realimentación del esquema de sincronización puede sufrir el mismo tipo de perturbación. En ambos casos, con ecuaciones de orden fraccionario o entero, construimos varios teoremas que aseguran que se alcance la sincronización en escenarios diversos, incluyendo, por ejemplo, casos en los que la matriz de acoplamiento de las redes es no difusiva (por lo tanto, podemos evitar esta hipótesis, que es ubicua en la literatura). En todos los casos de interés, mostramos que el esquema necesario para interconectar las redes es muy simple, puesto que se reduce al clculo de una nica matriz de ganancia cuya dimensión es independiente de la dimensión total (número de

nodos) de las redes. La estructura del sincronizados es también muy sencilla, lo que la hace potencialmente adecuada para aplicaciones del mundo real.

Aunque todos los esquemas que se proponen se analizan de manera rigurosa, también se muestran resultados numéricos (obtenidos mediante simulación) para ilustrar los métodos propuestos.

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Chapter 1

Introduction

1.1 Nonlinearity and chaos

A system whose state changes with time is often termed a "dynamical system" or "oscillator". If this change occurs in a continuous fashion over time, we refer to a continuous-time dynamical system, whereas we use the term "discrete time" for a dynamical system whose state remains constant for intervals of time and changes only at certain instants. In this thesis we focus on continuous time systems, the mathematical description of which is usually given by ordinary differential equations (ODEs) and a set of initial conditions, namely

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \\ x(t_0) &= x_0, \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is an $n \times 1$ vector of real variables that fully describes the system state at time $t \in \mathbb{R}$ (and, hence, we refer to it as the state vector), f is a function $\mathbb{R}^n \to \mathbb{R}^n$, also called vector field, and $x_0 \in \mathbb{R}^n$ is the initial condition, i.e., the value of the state vector at the initial time t_0. If the function f is nonlinear, then the system is said to be nonlinear, and it is termed linear otherwise. We define the dimension of a nonlinear system as the size of its state vector (number n of coordinates).

Chaos is a special property appearing in some nonlinear dynamic systems. Many biological, social, as well as electrical and mechanical systems exhibit chaotic behavior. Chaotic systems are special cases of general nonlinear systems, i.e., a linear system can never be chaotic. This means that all definitions and theorems that hold for nonlinear systems can be applied to chaotic systems. Chaotic systems are extremely sensitive to the value of the initial conditions and, although they can be described by fairly simple equations, they exhibit a very complex and random-like behavior.

The sensitivity to initial conditions that characterizes chaos also gives rise to a number of special, sometimes counter-intuitive, properties peculiar to chaotic systems. The study of such properties, often related to the synchronization and the control of chaotic oscillators, has attracted a great deal of attention and originated a whole field of investigation in the intersection of mathematics, physics and engineering. In Section 2.1.2 we provide an account of some of the more relevant of these properties.

1.2 Fractional-order dynamics

Fractional calculus as an extension of ordinary calculus has a 300 year old history. It has been found that the behavior of many physical systems can be properly described by using fractional-order system theory and important fundamental physical considerations in favor of the use of fractional-derivative based models are given, e.g., in [94]. Indeed, fractional-order derivatives provide a powerful instrument for the description of memory and hereditary properties of different substances. This is claimed to be the most significant advantage of the fractional-order models in comparison with integer-order models, which, in fact, neglect such effects.

A fractional-order differential equation can be written as

$$\begin{aligned} \frac{d^q x(t)}{dt^q} &= f\left(x(t)\right) \\ x\left(t_0\right) &= x_0, \end{aligned} \tag{1.2}$$

which is comparable to the ordinary system (1.1), except that the derivative degree q can take values between 0 and 2, $0 < q < 2$. Further details on fractional-order dynamics and its characteristic features are provided in Section 2.7.

A number of fractional-order systems exhibiting chaotic behavior have been identified by several authors [14, 15, 101, 66], a fact that has opened new paths for research.

1.3 Complex networks

After the seminal work in [103], complex network models have become ubiquitous in the analysis of many phenomena appearing in the physical, biological, and social sciences. The Internet and the World Wide Web, social networks, networks describing the interaction between cell components, and neural networks are some other samples of complex network systems around us.

Another feature of complex systems is that they exhibit properties that appear as a result of the local interactions between many of its constituent elements [24]. Remarkably, the emergent patterns are not present at a local scale, i.e., they are not inherent to the individual components, but rather emerge at the system level, due to the aggregate interactions [6].

The study of the topology of interactions in a large variety of real-world systems, in various fields, reveals that, despite the inherent differences, most complex networks are characterized by the same topological properties. This makes complex networks radically different from regular lattices and random graphs, the standard models studied in mathematical graph theory. *Small-world* networks [9, 114, 5, 113] and *scale free* networks [9, 19, 122] could be mentioned as the best known types of network models that appear to provide an adequate representation of a large number of real-world complex systems. A classification of complex network models together with a further discussion of their use in various scientific fields can be found in [109, 24]. For obvious reasons, involving the richness of their dynamical behavior, the study of networks composed of chaotic elements has become a prominent field of research in the last decade [9, 89, 113].

1.4 Network synchronization

With the pioneering work of Pecora and Carroll [91], it was shown that chaotic systems can be synchronized. Synchronization happens between two systems coupled in a proper way, such that using some feedback signals one of the systems is forced to mimic the trajectory of the other one. The phenomenon of synchronization between two dynamical systems is fundamental in science [23] and has a wealth of applications in technology [50].

In recent years, the study of synchronization phenomena in complex dynamical networks has attracted the interest of many researchers. Indeed, complex networks have become a mainstream area of research for over a decade, as they have been identified as powerful tools for modeling a variety of real-world systems that otherwise appear intractable.

In general, two kinds of network synchronization can be defined. The first (historically) one is the the synchronization of all nodes inside a network, which is often referred to as "inner synchronization" [9]. More recently, the possibility to synchronize two (initially separate) networks that are coupled in a suitable manner through some of their nodes has drawn attention from several authors [64, 28, 123]. This type of synchronization is usually referred to as “outer synchronization”, a term originally coined in [64]. In this thesis we focus on the study of the outer synchronization between two complex networks that are known up to some uncertainty, e.g., in the model parameters or in the feedback signals used in the coupling scheme.

The spread of an infectious disease across different groups of individuals is an example of a real-world phenomenon that can be modeled by way of outer synchronization. For example, the avian influenza was known to spread among domestic and wild birds, but at a later stage infected human beings unexpectedly [102].

Several other important examples can be found in the literature [111, 64]. We believe that the challenge of understanding the dynamics of coupled complex networks nowadays is one of the most relevant across various fields of science.

1.5 Robust control

The development of robust control theory began in the late 1970s and early 1980s and soon produced a number of techniques for dealing with bounded system uncertainty [21, 131]. Robust control methods are designed to function properly as long as uncertain parameters or disturbances are within some prescribed bounds. Therefore, they aim to achieve resilient performance and/or stability in the presence of bounded modeling errors.

Uncertainty is an unavoidable difficulty in any real world problem [21]. In general, two main facts make a model uncertain. The first one is that analytical or computational models which closely describe physical systems are difficult or impossible to precisely characterize and simulate. Whichever model we may propose, no matter how detailed, can never be a completely accurate representation of a real physical, biological or social system [130, 117, 128]. Even the model of a simple circuit could be uncertain, if we admit that the impedance of the resistors is subject to change by increasing the temperature, as a result of transmitting an electrical current through it [86]. Considering a time-variant resistance would reduce this uncertainty. However, trying to remove the uncertainty by modelling is typically a poor solution. On one hand, a model that was originally simple may easily become undesirably complex. On the other hand, each time we extend a model we can reasonably expect to extend the uncertainty as well.

In the example of the resistance, we may reckon that it will change with temperature, but the actual relationship can only be known approximately. Therefore, one enters an endless loop of modelling, identifying uncertainties and extending the model and, at each stage, the model becomes more intricate. Instead of trying to model a real-world system exactly, it is often more practical to represent the uncertainty in the model in a way that it remains simple and realistic. We may, for example, define some nominal values for the model parameters and then admit some variations around these values, variations often referred to as disturbances or perturbations. If the arbitrary parameter a has the nominal value a_n, the perturbed value $\tilde{a}$ can be defined as

$$\tilde{a} = a_n + \Delta a(t),$$

where $\Delta a(t)$ is the additive disturbance that is commonly considered to have an unknown but bounded value.

The existence of modeling errors is not the only fact leading us to robust control, however. Indeed, real systems are always exposed to environmental disturbances, or maybe implemented in unknown environments. So even if the model is extremely accurate, we may not be able to measure accurately all the inputs to the system. The effects of such uncertainties arising from external sources can also be mitigated by robust control methods.

Let us remark that robust control techniques rely on bounding the uncertainty rather than expressing it in the form of a probability distribution. A controller that can operate under bounded uncertainties is called a robust controller. Therefore, a controller is robust when a certain performance can be guaranteed with any choice (within a predefined set) of some parameters and functions involving the descriptive equations of the system dynamics. From this viewpoint, robust control theory might be stated as a worst-case analysis method rather than a typical case method. It must be recognized that some performance may be sacrificed in order to guarantee robustness. For example, one may have to generate control signals with higher energy (compared to an ideal case with no uncertainty).

1.6 Contributions

1.6.1 Summary

We have tackled the problem of outer synchronization between two complex networks in a master-slave configuration, whose constituent nodes are described by nonlinear differential equations, possibly leading to chaotic dynamics. An outline of the key contributions of this work is given below. Then we elaborate further on the specific results obtained for fractional-order and integer-order dynamical networks in Chapters 3 and 4, repectively.

- **Fractional-order dynamical networks.** We ha carried out the first (to our best knowledge) rigorous analysis of outer synchronization for complex networks described by fractional-order differential equations. It is based on the linearization of the synchronization error and, hence, it only guarantees local convergence. The results, however, are very broad in scope. They include networks described by integer-order differential equations as a special case, account for generalized synchronization phenomena and depend on very mild assumptions.

- **Global outer synchronization.** For the case of networks described by sets of integer-order differential equations, we have devised schemes that guarantee synchronization independently of the initial conditions of the system amd, again, under very mild assumptions regarding the network structure.

- **Robust synchronization.** Both for fractional and integer-order dynamics, our analysis is general enough to take into account modeling errors such as mismatches in the network parameters. Also, it naturally provides guidelines for the design of the controllers that ensure outer synchronization. The resulting schemes are non fragile as well, i.e., they are also robust to mismatches in the parameters of the controllers.

- **Synchronization schemes independent of the network size.** Based on a result on the eigendecomposition of certain Kronecker products of matrices, the construction of the synchronization schemes proposed in Chapters 3 and 4 is made independent of the network size. As a consequence, the computations required for the design of practical synchronizers are considerably simpler compared to other methods in the literature [64, 65, 69].

- **Mild assumptions on the network topology.** Most previous contributions from other auhors rely on a number of hypotheses on the coupling matrix that describes the topology of the networks (diffusivity, balance, symmetry, ...). Our approach allows to remove most of these assumptions (in some cases all of them).

1.6.2 Fractional-order networks dynamics

In Chapter 3, we tackle the outer synchronization of two perturbed complex networks with fractional-order dynamics. We first obtain a linearized version of the synchronization error dynamics and then carry out a stability analysis that provides simple sufficient and necessary conditions for the synchronization error to converge locally toward zero. Our approach avoids the need to compute eigenvalues of large system matrices (only their relative position is relevant) or to impose restrictive assumptions on the structure of the coupling matrices of the networks. Although we state our main results for the case of two identical networks with known parameters, we also show how they can be extended to systems in which the network parameters are perturbed. This extension is based on an alternative formulation of the conditions for the convergence of the synchronization error in terms of LMI's. Under some assumptions on the coupling matrices, we also provide analytical results regarding the generalized synchronization of the networks.

1.6.3 Integer-order networks dynamics

In Chapter 4, we investigate robust schemes for global outer synchronization of two perturbed complex networks. A theorem that provides a sufficient condition for the global outer synchronization of two networks with known parameters is proved. The sufficient condition in the latter theorem is formally given as an LMI that has to be satisfied by the system of coupled networks. The argument of the proof includes the design of the gain of the synchronizer, which is a constant square matrix with dimension given by the number of dynamic variables in a single network node. Therefore, the complexity of the scheme is independent of the size of the overall network, which can be much larger.

The basic result is subsequently elaborated, first in order to simplify the design of the synchronizer while holding the assumption of the coupling matrix being diffusive. Then, the latter assumption is relaxed and a sufficient condition for global outer synchronization is given. The corresponding LMI involves the maximum eigenvalue of the coupling matrix but avoids any other assumptions on it (in particular, the coupling matrix is not assumed to be diffusive anymore). Next, we investigate schemes that reduce the dimension of the synchronizer signals, which can be made lesser than the dimension of the state in a single node. Finally, we obtain synchronizers that are robust to model errors in the parameters of the networks. As before, sufficient conditions for global synchronization are given in the form of an LMI with only mild assumptions on the coupling matrix.

1.7 Thesis organization

The rest of the thesis is organized as follows.

In Chapter 2, we present the necessary mathematical background required to understand the rest of the thesis. This chapter opens with an introduction to chaos theory. We then provide basic concepts, definitions and stability theorems of nonlinear systems, followed by an introduction to linear matrix inequalities. These two topics are our most important tools in proving the main theorems. The problem of synchronization in chaotic systems, which is the base of our research, is then presented. We later discuss the concept of fractional-order differential equations, which is the governing dynamics of one of our two studied problems. After presenting the concept of complex network models, we are ready to define the main problem of this thesis, which is the synchronization between two complex networks.

In Chapter 3, the outer synchronization between two fractional-order networks is investigated. After the definition of networks with fractional-order dynamic nodes and the synchronization error, a key lemma on the eigenvalues of a certain class of matrices is introduced. Based on this lemma, the stability criterion of linear fractional-order systems and some mathematical manipulations, we produce a condition that guarantees the synchronization between two networks. We convert these conditions into the form of linear matrix inequalities, which facilitates the design of controllers and the subsequent analysis of networks with uncertain parameters. All the analytical results stated in this chapter refer to the *local* synchronization of the networks.

In Chapter 4, we study the outer synchronization between two networks with integer-order dynamical nodes. Besides the key lemma introduced in Chapter 3, we fruitfully use the different statements of the Lyapunov stability theorem to provide a condition that guarantees the global synchronization between the networks independently of their initial conditions. We then extended these results to systems with unknown parameters and external disturbances. Finally, we explore the possibility of synchronization with a reduced number of control channels between nodes of the two networks.

Chapter 5 contains a summary of the main results in this thesis, and suggests some ideas for future research in this field.

Chapter 2

Background

In this Chapter we introduce basic material to be used in the rest of this chapter. We start with the definition of chaotic dynamics and and its basic mathematical properties. Then we review the fundamentals of stability analysis and control of general nonlinear dynamical systems, which naturally leads to the introduction and discussion of linear matrix inequalities, a tool that is used extensively in subsequent chapters. The last sections of the chapter are devoted to the synchronization of chaotic systems, later extended to complex networks, and, finally, an introduction to dynamical systems described by fractional order differential equations.

2.1 Chaotic dynamical systems

The theory of chaos is one of those mathematical and physical constructs that can instantaneously seize the people's imagination and interest. It transcends the disciplines: philosophy, religion, mythology and science each has its own perspective on chaos. In this section, we give some mathematical insight and facts on the theory of chaos. We start with a historical survey that lays the grounds of what is known today as nonlinear science and chaos theory. Then, we introduce the most relevant properties of chaotic systems.

2.1.1 Historical perspective

In ancient Greek mythology, chaos was the "primeval emptiness preceding the genesis of the universe, turbulent and disordered, mixing all the elements" (adapted from [119]). From this turmoil, order eventually emerged to shape the world. Though naive, this tale connects two key concepts of the modern theory of chaos and makes them interdependent: order and disorder. Philosopher Aristotle also articulated an important property that characterizes chaos, and will be later known as the sensitivity to initial conditions (SIC). The conclusion he drew was that “the least initial deviation from the truth is multiplied later a thousandfold” [10]. With this statement, Aristotle described a form of exponential divergence with time; a slightly modified (one could say disturbed) original concept or “truth” may end with a complete different and unexpected final form.

Finding its roots in social sciences and Greek myth, the ideas of chaos and SIC were considered irrelevant from a scientific point of view for centuries. Only in 1876, as James Clerk Maxwell was developing his kinetic theory, he argued that a small variation in the current state makes the prediction of future states impossible. At this time, however, he was convinced that the key factor rendering his effect visible was the complexity of the system through its large number of variables. Later in 1892, the problem of stability was addressed mathematically by the Russian mathematician Aleksandr Lyapunov. For the first time, he calculated the divergence rate between the trajectories of identical dynamical systems with different initial conditions. At about the same time, in 1898, the French mathematician Jacques Hadamard remarked that a discrepancy in the initial conditions of a dynamical system could lead to an unpredictable long-term evolution. In 1908, another French mathematician, Henri Poincarè, deepened Hadamard’s idea and concluded that any prediction of future states was impossible, as a result of his famous study of the stability of the 3-body problem.

Other significant milestones in the theory of dynamical systems came about after Henri Poincarè discoveries. We cite the work of B. Van der Pol and Aleksander Andronov in the 1920’s and 1930’s on the study of oscillations in relaxed and self-sustained oscillators, respectively. In the 1950’s, Kolmogorov, Arnold, and Moser focused their attention on the persistence of motion of quasi-periodic oscillators and obtained the fundamental KAM Theorem [1].

In the 1960’s, the theory of chaos received unprecedented attention as Edward Lorenz, a meteorologist at the Massachusetts Institute of Technology (MIT), proposed a graphical representation of SIC in a simplified numerical model of the Earth atmosphere. The model he developed is a 3-

[1] The KAM Theorem proves the existence of invariant tori (quasi-periodic trajectories) in the phase space of an integrable hamiltonian system after perturbation [12, Appendix 8].

dimensional nonlinear model described by

$$\begin{cases} \dot{x}_1(t) = -\sigma\left(x_1(t) - x_2(t)\right) \\ \dot{x}_2(t) = \rho x_1(t) - x_2(t) - x_1(t)x_3(t) \\ \dot{x}_3(t) = -\beta x_3(t) + x_1(t)x_2(t) \end{cases}, \tag{2.1}$$

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are dynamic variables, while σ, ρ and β are static static parameters. Lorenz wanted to analyze data produced by his model on long sequences; however, at this time computing power was extremely limited. Therefore, to obtain large sequences, one had to run multiple sequential simulations. It is precisely what he did, except that when he initiated the next simulation with the last results from the previous run with a lower precision, he noticed that the model did not duplicate the expected evolution that a single simulation would have produced (see Figure 2.1).

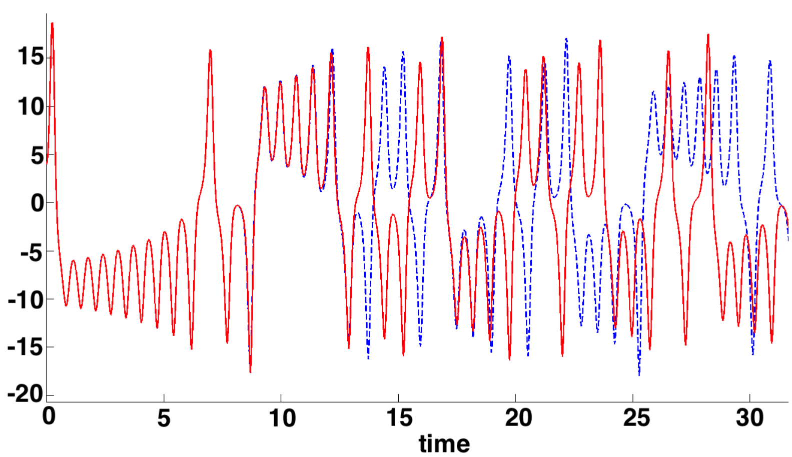

Figure 2.1: Numerical evidence of the sensitivity to initial condition in the Lorenz system, as observed historically by Lorenz. Depicted in blue dashed line is the evolution with initial condition with 5-digit precision; depicted in red solid line is the same evolution with a duplicated initial condition with a 3-digit precision.

Contrary to his expectations, the lower-precision initial conditions would not have negligible consequences on the system's dynamics. This discovery and subsequent work contributed to explain the inaccuracy of long-term weather forecasting and were summarized by E. Lorenz at the 139th meeting of the American Association for the Advancement of Science (AAAS) with this now famous statement: “Does the flap of a butterfly’s wings in Brazil set off a tornado in Texas?” [71]. That is how the SIC was also known as the “butterfly effect”. After this major turn, research on nonlinear dynamics and chaos theory stepped up.

In 1971, David Ruelle and Floris Takens proposed an alternative mathematical explanation of the turbulence in fluid dynamics based on the existence of so-called “strange attractors” [97]. A couple of years later, Tien-Yien Li and James A. Yorke used the term chaos to describe the erratic and unpredictable behavior arising in deterministic nonlinear maps. At about the same time, Mitchell J. Feigenbaum unraveled the universality of behavior occurring in a particular class of systems as they transition to chaos, and derived the Feigenbaum constant [43].

2.1.2 Properties of chaotic systems

In this section, some special properties of chaotic systems are brought.

Sensitivity to initial conditions (SIC) [104]: SIC can be considered as the most apparent property of chaotic systems, and as such it caused the discovery by Lorenz, as mentioned in the previous section. In a chaotic system, two arbitrarily close points in the state space lead to significantly different future trajectories. Thus, an arbitrarily small perturbation of the current trajectory may lead to significantly different future behavior. Numerical evidence of SIC in the Lorenz system is shown in Figure 2.1.

Strange attractors [104]: An attractor is a set towards which a dynamic variable evolves over time. An attractor can be a point, a finite set of points, a curve, a manifold, or even a complicated set with a fractal structure known as a strange attractor. Having strange attractors is a peculiar property of chaotic systems. Figure (2.2) shows the strange attractors of the Lorenz system.

Some other properties of chaotic systems are listed below. Since we do not make an explicit use of them later chapters, short description suffices.

Topological mixing [104]: Topological mixing (or topological transitivity) means that the system evolves over time so that any given region or open set of its phase space eventually overlaps with any other given region.

Density of periodic orbits [104]: Every point in the space is approached arbitrarily closely by periodic orbits.

Minimum complexity of a chaotic system [104]: Discrete-time chaotic systems, such as the logistic map, can exhibit strange attractors whatever their dimensionality. In contrast, for continuous-time dynamical systems, the Poincarè Bendixson theorem shows that a strange attractor can only arise in dimensions three or higher.

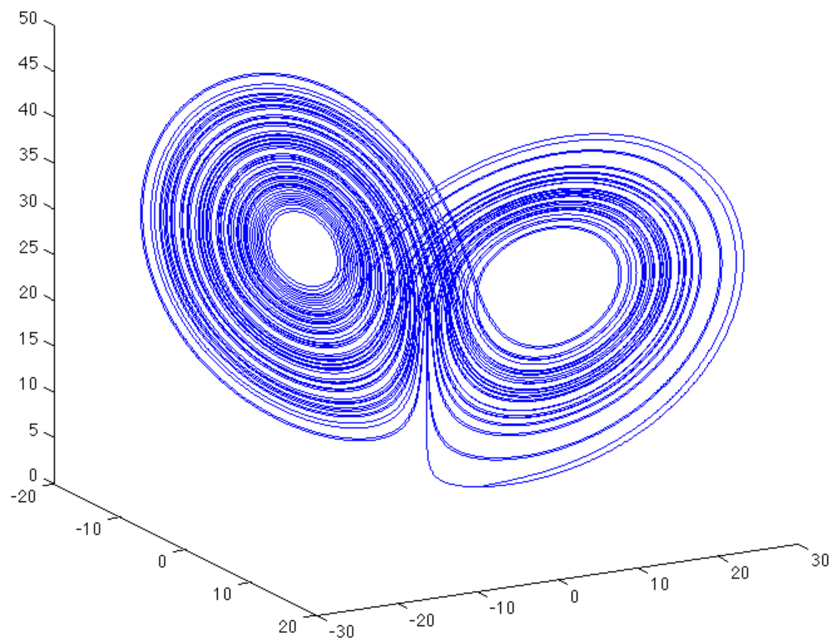

Figure 2.2: Strange attractor of the Lorenz system.

2.2 Stability of nonlinear systems

The concept of stability is ubiquitous in dynamical system theory and it underlies the notions of attractors, bifurcation theory, and synchronization. We first need to define the equilibrium point of a dynamic system. Consider a general nonlinear system defined by

$$\begin{aligned} \dot{x}(t) &= f\left(x(t)\right) \\ x\left(t_0\right) &= x_0, \end{aligned} \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ is the state space with dimension n at time t, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function and $x_0 \in \mathbb{R}^n$ is the initial condition.

For system (2.2), the vector $x(t) = x_e$ is an equilibrium point if

$$f(x_e) = 0.$$

It is possible to define different kinds of stability. The two main types are described below.

Definition 2.1 *An equilibrium point is stable in the Lyapunov sense if for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\text{if} \quad \|x\,(t_0)\,\| < \delta \quad \text{then} \quad \|x\,(t)\,\| < \epsilon, \quad \forall t > t_0$$

Lyapunov stability guarantees that the trajectory of the system in phase space will remain in a certain vicinity of the equilibrium point, as long as the initial state belongs to a region $\mathcal{D}$ defined by $\|x\,(t_0)\,\| < \delta$. This kind of stability guarantees that the states of the system do not diverge. In practice, we usually need to show the the states of (for example) an error system converge to zero. This is given explicitly in the next definition.

Definition 2.2 *An equilibrium point is asymptotically stable if there exist $\delta > 0$ such that*

$$\|x\,(t_0) - x_e\| < \delta \quad \text{implies} \quad \lim_{t\to\infty} \|x\,(t) - x_e\| = 0,$$

where t_0 is the initial time instant.

Asymptotic stability includes Lyapunov stability as a particular case, but imposes that all trajectories initiated in the neighborhood of the equilibrium point must converge asymptotically towards it. Furthermore, a system is globally asymptotically stable if $\lim_{t\to\infty} x(t) = 0$ for all initial states $x(t_0)$.

A major inconvenient with the definition asymptotical of stability is that it requires the evaluation of the state trajectories. However, it is much easier, and also practical, to check the stability of a system without necessity of making the system run. This is precisely what Lyapunov stability theorems provide.

Lyapunov stability is named after Aleksandr Lyapunov, a Russian mathematician who published his book “The General Problem of Stability of Motion” in 1892 [77]. His work, initially published in Russian and then translated into French, received little attention for many years. Interest in it started in the 1950's when the so-called "Second Method of Lyapunov" (see below) was found to be applicable to the stability of aerospace guidance systems which typically contain strong nonlinearities not tractable by other methods. A large number of publications appeared ever since in the control and systems literature.

Lyapunov, in his original 1892 work, proposed two methods for proving stability [77]. The first method developed the solution in a series which was then proved to converge within certain bounds. The second method, which is almost universally used nowadays, makes use of a Lyapunov function $V(x)$ which has an analogy with the potential function of classical dynamics. Theses two methods are described below.

Lyapunov first method (indirect method): The indirect method of Lyapunov uses the linearization of a system to determine the local stability of the original system. Consider the system (2.2) with continuously differentiable f, where $f(0) = 0$. Define

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} \tag{2.3}$$

to be the Jacobian matrix of $f(x)$ with respect to x, evaluated at the origin. Then, the system

$$\dot{z}(t) = Az(t) \tag{2.4}$$

is referred to as the linearization of equation (2.2) around the origin. The stability of system (2.4) determines the local stability of the original nonlinear equation. If matrix $A(t)$ in (2.4) is time-invariant, then we can check the stability by means of the following lemma.

Lemma 2.1 *[59] The linear time-invariant system*

$$\dot{z}(t) = Az(t) \tag{2.5}$$

with state vector $z(t) \in \mathbb{R}^n$ and system matrix A is globally asymptotically stable if, and only if, all eigenvalues of matrix A are in the open left half complex plane. Moreover, the system is stable if all eigenvalues on the imaginary axis (if any) have order one.

Lyapunov second method (direct method): In this method no linearization is done, but the actual system itself is is analysed. Unlike with the indirect method, the stability analysis by the direct method is global, not local. This method, which is almost universally used nowadays, is a generalization of the idea that if there is some "measure of energy" in a system then we can study the rate of change of the energy of the system to ascertain stability.

Theorem 2.1 *[59] Let $x = 0$ be the equilibrium point for (1.1) and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function, such that*

$$\begin{aligned} V(0) &= 0 \text{ and } V(x(t)) > 0 \quad in \quad D \backslash \{0\} & (2.6) \\ \dot{V}(x(t)) &\leqslant 0 \quad in\, D & (2.7) \end{aligned}$$

Then $x = 0$ is stable. Moreover, if

$$\dot{V}(x(t)) < 0 \quad in\, D \backslash \{0\} \tag{2.8}$$

then $x = 0$ is asymptotically stable

Theorem 2.1 gives sufficient conditions for the stability of the origin of a system. It does not, however, give a prescription for determining the Lyapunov function $V(x(t))$. Since the theorem only gives sufficient conditions, the search for a Lyapunov function establishing stability of an equilibrium point could be arduous. However, it is a remarkable fact that the converse of Theorem 2.1 also exists: if an equilibrium point is stable, then there exists a function $V(x(t))$ satisfying the conditions of Theorem 2.1. However, the utility of this and other converse theorems is limited by the lack of a computable technique for generating Lyapunov functions. Theorem 2.1 also stops short of giving explicit rates of convergence towards equilibrium. It may be modified to do so in the case of exponentially stable systems.

Definition 2.3 *The equilibrium point $x^* = 0$ is an exponentially stable equilibrium point of (1.1) if there exist constants $m, \alpha > 0$ and $\epsilon > 0$ such that*

$$\|x(t)\| \leqslant m e^{-\alpha(t-t_0)} \|x(t_0)\| \tag{2.9}$$

for all $\|x(t_0)\| \leqslant \epsilon$ and $t \geqslant t_0$. The largest constant α which may be utilized in (2.9) is called the rate of convergence.

Exponential stability is a strong form of stability. Exponential convergence is important in applications because it can be shown to be robust to perturbations and is essential for the consideration of more advanced control algorithms, such as adaptive ones [59]. A system is globally exponentially stable if the bound in equation (2.9) holds for all $x_0 \in \mathbb{R}^n$.

Theorem 2.2 *[59]* $x^* = 0$ *is an exponentially stable equilibrium point of* $\dot{x}(t) = f(x(t))$ *if, and only if, there exists* $\epsilon > 0$ *and a function* $V(x)$ *which satisfies*

$$\begin{aligned} \alpha_1 \|x\|^2 \leqslant \quad V(x) \quad &\leqslant \alpha_2 \|x\|^2 \\ \dot{V}(x) \quad &\leqslant -\alpha_3 \|x\|^2 \\ \left\| \frac{\partial V}{\partial x}(x) \right\| \quad &\leqslant \alpha_4 \|x\| \end{aligned} \tag{2.10}$$

for some positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$*, and* $\|x\| \leqslant \epsilon$*.*

It can be shown that Theorem 2.2 introduces bounds on m and α used in (2.9), namely

$$m \leqslant \left(\frac{\alpha_2}{\alpha_1} \right)^{1/2} \quad \text{and} \quad \alpha \geqslant \frac{\alpha_3}{2\alpha_2}. \tag{2.11}$$

The Lyapunov theory has been one of the most effective tools in the control of dynamical systems. This claim is evidenced by the fact that, although classical and well established in the literature, this concept is even today extensively exploited in many different practical and theoretical problems. In particular, the Lyapunov approach is one of the most powerful tools to deal with the problem of controlling uncertain systems. This theory is practically necessary when dealing with uncertain (especially nonlinear) systems with time-varying parameters. Moreover, for important classes of problems and special classes of functions the theory is supported by efficient numerical tools such as those based on LMI's. We extensively use the Lyapunov stability criteria in LMI structure in this thesis.

2.3 Linear matrix inequalities

In this section, we introduce linear matrix inequalities (LMI's). LMI's appear in many control problems, such as the Positive Real lemma, quadratic optimization problems, $\mathcal{H}^\infty$ design problems, and so on. There exist very reliable numerical solution tools for LMI. In this thesis, we have used the *yalmip* toolbox in Matlab.

2.3.1 Basic definitions

Basic definitions of matrix properties, as well as the definition of LMI are provided here.

Definition 2.4 *The $n \times n$ matrix P is positive definite if for all nonzer vectors $u \in \mathbb{R}^n$, we have*

$$u^\top P u > 0.$$

Matrix P is *negative definite* if $u^\top P u < 0$. Replacing the signs $>$ and $<$ with $\geqslant$ and $\leqslant$ gives the definitions of *semi-positive definite* and *semi-negative definite* matrix, respectively.

Definition 2.5 *For the $n \times n$ matrix P, the following notations are defined.*

$$P > 0: \quad P \text{ is positive definite,} \tag{2.12a}$$

$$P \geqslant 0: \quad P \text{ is semi-positive definite,} \tag{2.12b}$$

$$P < 0: \quad P \text{ is negative definite,} \tag{2.12c}$$

$$P \leqslant 0: \quad P \text{ is semi-negative definite.} \tag{2.12d}$$

With these preliminaries, we can provide the definition of an LMI.

Definition 2.6 *An LMI has the form*

$$F(x) \triangleq F_0 + \sum_{i=1}^{n} x_i F_i > 0, \tag{2.13}$$

where $x_i \in \mathbb{R}^n$, $i = 1, \cdots, n$, are variables and the symmetric matrices $F_i = F_i^\top \in \mathbb{R}^{n\times n}$, $i = 0, \cdots, n$, are given.

We say this LMI is *feasible* if there exists a set of solutions $\{x_i \in \mathbb{R}^n$, $i = 1, \cdots, n\}$, that satisfy (2.13).

2.4 Synchronization of chaotic systems

2.4.1 Historical Perspective

Synchronization comes from the greek words “syn” (with) and “chronos” (time), literally occurring at the same time. Synchronization of oscillators is a universal and ubiquitous phenomenon in nature [92]. It was discovered by Christian Huygens in 1665, who observed perfect in- and out-of-phase oscillations of two pendulums clocks dynamically coupled by their common support and concluded on the existence of “sympathy on two clocks” [53].

The synchronization of chaotic systems was long thought to be counterintuitive or impossible, especially because of the sensitivity to initial conditions preventing two identical chaotic systems from displaying perfectly correlated time evolutions. However, in 1983, Fujisaka and Yamada paved the way with their pioneering studies on chaos synchronization [45, 125, 126] followed by the work of L. Pecora and T. Carroll, who demonstrate theoretically and experimentally the existence of complete synchronization with an electronic version of a Lorenz system [91].

2.4.2 Basic definitions

There exist various types of synchronization. We propose in this subsection a non-exhaustive rapid overview of their mathematical formulations. In this thesis we only deal with complete and generalized synchronization, albeit in the context of complex networks.

We define the chaotic master and slave systems as

$$\dot{x}(t) = f(x), \quad \dot{y}(t) = g(y) + u, \tag{2.14}$$

respectively, where $x \in R^n$ and $y \in R^n$ are state vectors and u is the synchronizer signal.

According to (2.14), different types of synchronization can be defined as follows.

Definition 2.7 *In* ***complete synchronization****, the states of the interacting systems x and y converge asymptotically to the same trajectory, i.e.:*

$$\lim_{t \to \infty} \|x(t) - y(t)\| = 0. \tag{2.15}$$

Complete synchronization was originally described in [22].

Definition 2.8 *In* ***generalized synchronization****, the states of the two interacting systems are functionally related. There exists a function $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\lim_{t \to \infty} \|x(t) - \Psi\left(y(t)\right)\| = 0. \tag{2.16}$$

This type of synchronization was introduced for the first time in [98].

In most cases described so far, the interaction between the systems is instantaneous. In practice and particularly in the optoelectronic systems, the interactions are delayed. This leads to the definition of anticipating synchronization.

Definition 2.9 *Anticipating synchronization is established when*

$$\lim_{t\to\infty} \|x(t) - y(t-\tau)\| = 0. \tag{2.17}$$

with τ the time delay.

This type of synchronization was proposed for the first time in [110].

For a detailed treatment of the synchronization of nonlinear systems, we refer the reader to Ref. [23].

2.4.3 Properties of LMI's

A very useful property of LMI is convexity. In simple words, convexity means that if the LMI's

$$\underline{F_0} + \sum_{i=1}^{n} x_i \underline{F_i} > 0, \tag{2.18a}$$

$$\overline{F_0} + \sum_{i=1}^{n} x_i \overline{F_i} > 0 \tag{2.18b}$$

with a solution set $\{x_i^* \in \mathbb{R}^n,\ i = 0, \cdots, n\}$ are **simultaneously** satisfied, then we have

$$F_0 + \sum_{i=1}^{n} x_i^* F_i > 0, \tag{2.19}$$

where

$$F_i = \lambda_i \underline{F_i} + (1-\lambda_i)\,\overline{F_i}, \quad \lambda_i \in [0,1]\,, \quad i = 0, \cdots, n. \tag{2.20}$$

As we see in next chapters, this property is very useful when the system uncertainties can be mapped to a convex area. The following lemma is a very useful tool in connection with LMI's. It can convert a special, but common, family of bilinear matrix inequalities into an LMI structure.

Lemma 2.2 ***(Schur Complement)*** *Let P be a symmetric matrix defined by*

$$\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}; \tag{2.21}$$

then we have

$$X > 0 \quad \textit{if, and only if,} \quad A > 0,\ C - B^\top A^{-1} B > 0$$

and

$$X > 0 \quad \textit{if, and only if,} \quad C > 0,\ A - B^\top C^{-1} B > 0.$$

2.4.4 Some examples

The history of LMI's in the analysis of dynamical systems goes back more than 100 years. The story begins circa 1890, when Lyapunov showed that the system (2.5) is stable if and only if there exists a positive definite matrix P such that the LMI (2.22) is satisfied[2]. To be specific, the following result can be proved.

Lemma 2.3 *System* (2.5) *is exponentially asymptotically stable if, and only if, the following LMI is satisfied*

$$A^\top P + PA < -Q. \tag{2.22}$$

when $P > 0$ and $Q > 0$ are positive matrices with appropriate dimensions.

It is easy to show that (2.22) can be expressed in the form of (2.13) [26].

Another important LMI in control theory is the Riccati inequality. After the pioneering paper [118], algebraic Riccati equations have been used extensively in optimal control. Optimal controllers can be constructed by computing a positive definite symmetric matrix P that satisfies the algebraic Riccati inequality

$$A^\top P + PA + PBR^{-1}B^\top P + Q < 0, \tag{2.23}$$

where A and B are fixed matrices, Q is a fixed symmetric matrix, and R is a fixed symmetric positive definite matrix. The Riccati inequality is quadratic in P but can be expressed as an LMI by applying the Schur complement lemma, namely

$$\begin{bmatrix} -A^\top P - PA - Q & PB \\ B^\top P & R \end{bmatrix} < 0. \tag{2.24}$$

The Ricatti inequality also plays a role in checking the *passivity* of linear systems, which is a key property for developing robustness analysis tools for linear systems. It also appears in the *bounded real lemma* approach and checking the $\mathcal{H}_\infty$ of a transfer function. Needless to say, in each case the resulting Ricatti inequality may be apparently different, but it can be rewritten in the general form of (2.24). The interested reader is referred to [26].

[2] This theorem is actually equivalent to Theorem 2.1

2.5 Complex networks

Networks are all around us, and we are ourselves, as individuals, the units of a network of social relationships of different kinds and, as biological systems, the delicate result of a network of biochemical reactions [20, 84, 4]. Coupled biological and chemical systems, neural networks, social interacting species, the process of spreading a disease, the Internet and the World Wide Web, are some other examples of systems composed by a large number of highly interconnected dynamical units. Networks can be tangible objects in the Euclidean space, such as electric power grids, the Internet, highways or subway systems, and neural networks. Or they can be entities defined in an abstract space, such as networks of acquaintances or collaborations between individuals.

A network description involves a reduction of the system's components to nodes and a reduction of the interactions between the components to links, connecting the nodes. Naturally, in a network description many details of the original system may be neglected. However, the simplification still captures the essential features and facilitates the mathematical analysis. For example, structural properties of the derived network can provide insight into both the regular functioning and the failure of the system under consideration, or allow the identification of critical and redundant components of the system.

Graph theory [25, 115] is the natural framework for the exact mathematical treatment of complex networks and, formally, a complex network can be represented as a graph. In general, a network can be defined as a set of nodes joined by some links. As mentioned before, such a definition is convenient to describe a variety of systems in many scientific fields, including biology, infrastructures, social systems, internet and others. For example, a scientific collaboration network would be made up by considering the scientists and co-authored papers as nodes and links of a network, respectively. The air transport network consisting of airports, as nodes, and air routes, as links, is another example.

In Figure 2.3, some general schematics of networks are brought forth. If no direction is defined for the links, the network is called undirected (Figure 2.3(a)). The example of a scientific collaboration network is an instance of an undirected network. The network is called directed when the connection between nodes has a flow direction (Figure 2.3(b)). For example, in a disease spread network, the links between two nodes (i.e., two sick persons) is defined as an arrow from the transmitter to the newly infected individual. In Figure 2.3(c), a weighted undirected network is shown. Existence of strong and weak ties between individuals in social networks [79, 83] and the diversity of

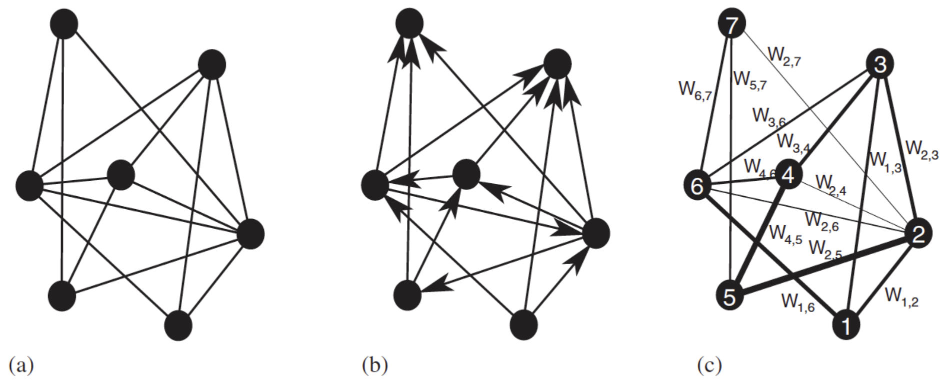

Figure 2.3: Graphical representation of a undirected (a), a directed (b), and a weighted undirected (c) graph with 7 nodes and 14 links. In the directed graph, adjacent nodes are connected by arrows, indicating the direction of each link. In the weighted graph, the values $w_{i,j}$ reported on each link indicate the weights of the links, and are graphically represented by the link thicknesses.

the predator-prey interactions in food webs [95] are two examples of weighted undirected and weighted directed networks, respectively.

2.6 Network synchronization

2.6.1 Mathematical representation of a network

In Section 1.3, the basic definition and scheme of a network were presented. Here, we provide the mathematical description of a network. Consider a network that consists of N identical nodes, each one being a dynamical subsystem described by an n-dimensional system of differential equations[3] as

$$\dot{x}(t) = Ax(t) + f(x(t)), \tag{2.25}$$

where $x_i\,(t) \in \mathbb{R}^n$ is the state vector at time t, comprising n real variables, $A \in \mathbb{R}^{n\times n}$ is an $n \times n$ matrix with real entries that describes the linear component of the node dynamics, while $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function that describes the nonlinear component of the dynamics of the system.

[3]The equation (2.25) is essentially the same as general presentation of a nonlinear system ($\dot{x}(t) = f(x)$), except that the difference that linear and nonlinear parts are separated.

Consider a network made by connecting nodes described by equations of the form of (2.25) together in a certain manner, described by the coupling matrix C. Such a network can be described as

$$\dot{x}_i(t) = Ax_i(t) + f\left(x_i(t)\right) + \sum_{j=1}^{N} c_{ij} L x_j(t), \quad i = 1, \dots, N. \tag{2.26}$$

The entries of the matrix $L \in \{0,1\}^{n \times n}$ are binary, linking scalar state variables in $x_i(t)$, while the $N \times N$ coupling matrix $C \in \mathbb{R}^{N \times N}$ indicates the network topology, i.e., the connections existing among the N nodes. We write c_{ij} to denote the (real) entry in the i-th row and j-th column of C. If $c_{ij} > 0$ then there exists a link from node i to node j ($i \neq j$), while $c_{ij} = 0$ when the nodes are not connected directly. If positive, the entry c_{ij} indicates also the strength of the connection[4] between the nodes i and j. Function f is assumed to be Lipschitz with constant η, i.e.,

$$\left\| f\left(x\right) - f\left(y\right) \right\| \leq \eta \left\| x - y \right\| \quad \forall x, y \in \mathbb{R}^n. \tag{2.27}$$

In Section 2.6.3, we discuss in further detail the different possible assumptions on matrix C and how they aid the design of synchronizer schemes.

2.6.2 Outer synchronization

We focus on the outer synchronization between two uncertain[5] complex networks. The reader is referred to [76, 81, 9] for an in-depth discussion of inner synchronization.

In order to investigate outer synchronization between two identical networks, we consider Eq. (2.26) as the master network and assume the response system to be coupled with the master through the scheme

$$\dot{y}_i(t) = Ay_i(t) + f\left(y_i(t)\right) + \sum_{j=1}^{N} c_{ij} L y_j(t) + u_i(t) \tag{2.28}$$

where $y_i(t) \in \mathbb{R}^n$ is the $n \times 1$ state vector of the response network, $u_i(t) \in \mathbb{R}^n$ is the synchronizer signal, defined as $u_i = K(y_i - x_i)$, and K is a constant matrix which will be later designed in such a way that outer synchronization

[4] According to the different types of coupling described in Section 1.3, we here consider the most general one, namely weighted directed coupling.

[5] The uncertainty in this context refers to mismatches in the network parameters and/or any unknown perturbations of the system dynamics (e.g., additive noise processes).

can be guaranteed. To be specific, we define global outer synchronization between the networks defined by Eqs. (2.26) and (2.28) as follows.

Definition 2.10 *Networks* (2.26) *and* (2.28) *synchronize globally if*

$$\lim_{t\to\infty} \|y_i(t) - x_i(t)\| = 0, \quad \textit{for} \quad i = 1, 2, \ldots, N. \tag{2.29}$$

This definition clearly is an extension of complete synchronization as defined in Section 2.4.2. Other types of outer synchronization can defined in the same manner. We also investigate general synchronization between two fractional-order systems in Chapter 4.

2.6.3 Role of the coupling matrix

It has been reported that the network coupling may affect heavily on the synchronization scheme when some techniques are applied [64, 69, 28]. In general, matrix C is a key element for characterizing the dynamics of both Eqs. (2.26) and (2.28). In the literature, various assumptions are commonly made in order to simplify the analysis of the class of systems described by Eq. (2.26). Specifically, most authors assume the following properties.

Diffusivity: The matrix C satisfies $\sum_{j=1}^{N} c_{ij} = 0$, $i = 1, 2, \ldots, N$. As a consequence, its diagonal elements can be written as $c_{ii} = -\sum_{j=1, j\neq i}^{N} c_{ij}$ (hence $c_{ii} < 0$, if the network is connected). This assumption is almost invariably made in the literature. Assuming diffusivity implies that all eigenvalues of matrix C have nonpositive real parts. According to the Theorem 2.1 on the stability of linear systems, this property is very useful, particularly for methods that consider the linearized approximation of the model.

Other usual assumptions on the coupling matrix are listed below.

Symmetry: This happens when the coupling matrix is symmetric, i.e., for all i, j, $c_{ij} = c_{ji}$. In a symmetric matrix, all eigenvalues are real.

Irreducibility: The network is connected in such a way that for any two nodes, there are always one or more links connecting them together. In other words, there are no isolated clusters of nodes. Some properties of irreducible matrices are described in [120].

Balance: Matrix C is balanced if

$$\sum_{j\neq i} a_{ij} = \sum_{j\neq i} a_{ji}, \quad i = 1, 2, \cdots, N.$$

This assumption may also facilitates the analysis of synchronization. In [69], for example, the authors consider the coupling matrix to be both diffusive and balanced, and they obtain the following equality

$$\sum_{j\neq i}\left(A+A^{\top}\right)=\sum_{j\neq i}(a_{ij}+a_{ji})=2\sum_{j\neq i}a_{ij}=-2a_{ii}.$$

In the next chapters, we see that one of the contributions of the current work is to relax some, and in some cases all, of these assumptions, even diffusivity.

2.7 Fractional order dynamics

2.7.1 Historical notes

The concept of the differentiation operator $D=\frac{d}{dx}$ is familiar to all those who have studied elementary calculus. For suitable functions, the nth derivative of f, namely $D^n f(x)=\frac{d^n f(x)}{dx^n}$, is well defined when n is a positive integer [93] .

On September 30th, 1695, L'Hopital asked Leibniz what meaning could be ascribed to $D^n f$ if n were a fraction. *"This is an apparent paradox from which, one day, useful consequences will be drawn"*. These words are Leibniz's response, what today is actually realized. Since that time fractional calculus has drawn the attention of many outstanding mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, and Laurent. But it was not until 1884 that the theory of generalized operators achieved such a level in its development so as to make it suitable as a starting point for the modern mathematicians [72].

By then the theory had been extended to include D^m operators, where m could be rational or irrational, and positive or negative.Thus the name fractional calculus became somewhat of a misnomer. A better description might be differentiation and integration to an arbitrary order. However, we shall adhere to tradition and refer to this theory as fractional calculus.

During the investigations of the general theory and applications of differintegrals (a term that was coined to avoid the cumbersome alternate "derivatives or integrals to arbitrary order"), it was discovered that, while this subject is old, dating back at least to Leibniz in its theory and to Heaviside in its application, it had been studied relatively little since the early papers which only hinted at its scope. Maybe the main reason was lack of today's high speed computers which provide the necessary capabilities to evaluate some related mathematical functions.

2.7.2 Real world applications

In the last years a revival of interest in the subject seems to have taken place [72, 129, 30], but the application of these ideas has not yet been fully exposed, primarily because of their unfamiliarity. Our studies have convinced us that differintegral operators may be applied advantageously in many diverse areas. Within mathematics, the subject is in contact with a very large segment of classical analysis and provides a unifying theme for a great number of well-known, and some new, results. Applications outside mathematics include otherwise unrelated topics such as: transmission line theory, chemical analysis of aqueous solutions, design of heat-flux meters, rheology of soils, growth of intergranular grooves on metal surfaces, quantum mechanical calculations, and dissemination of atmospheric pollutants.

Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [52]. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields. Fractional integrals and derivatives also appear in the theory of control of dynamical systems, when the controlled system or/and the controller is described by a fractional differential equation. The mathematical modeling and simulation of systems and processes, based on the description of their properties in terms of fractional derivatives, naturally leads to differential equations of fractional order and to the necessity to solve such equations.

The idea of fractional derivatives and integrals seems to be quite a strange topic, very hard to explain, due to the fact that, unlike commonly used differential operators, it is not related to some important geometrical meaning, such as the trend of functions or their convexity. For this reason, this mathematical tool could be judged “far from reality”. But many physical phenomena have an “intrinsic” fractional order description and so fractional order calculus is necessary in order to explain them. For example, fractional derivatives have been widely used in mathematical modeling of viscoelastic materials [16, 44, 96]. Some electromagnetic problems can be described by using fractional differ-integration [51, 41]. In physical chemistry, the current is proportional to the fractional derivative of the voltage when the interface is put between a metal and an ionic medium [62]. In the fractional capacitor theory, if one of the capacitor electrodes has a rough surface, the current passing through it is proportional to the non-integer derivative of its voltage [33]. Also, the existing memory in dielectrics used in capacitors is justified by fractional derivative based models [116]. The anomalous diffusion phenomena in inhomogeneous media can be explained by non-integer derivative based equations of diffusion [11, 40]. The electrode-electrotype interface is an example of fractional-order processes because at metal-electrolyte interfaces the impedance is proportional to the non-integer order of frequency for small angular frequencies [54, 57].

Another example for an element with fractional-order model is the fractance. The fractance is an electrical circuit with non-integer order impedance [61]. This element has properties that lie between resistance and capacitance. Tree fractance [82] and chain fractance [88] are two well known examples of fractances. The resistance-capacitance-inductance (RLC) interconnect model of a transmission line is a fractional-order model [29]. Heat conduction as a dynamical process can be more adequately modeled by fractional-order models than integer-order models [56]. In biology, it has been deduced that the membranes of cells of biological organism have fractional-order electrical conductance [32] and then are classified in the group of non-integer order models. Also, it has been shown that modeling the behavior of brainstem vestibule-oculumotor neurons by fractional-order differential equations has more advantages than classical integer-order modeling [7]. In mechanics, it has been found that the water flow on a dyke with porous internal structure is proportional to the fractional derivative of the dynamic pressure at the water/dyke interface [90]. In economy, it has been known that some finance systems can display fractional order dynamics [60]. More examples for fractional-order dynamics can be found in Ref. [100]. Moreover, applications of fractional calculus have been reported in many areas such as signal processing [78], image processing [31], automatic control [17] and robotics [42]. These examples and many other similar samples perfectly clarify the importance of consideration and analysis of dynamical systems with fractional-order models.

2.7.3 Preliminaries and definitions

The differintegral operator, denoted by D_t^q, is a combined differentiation-integration operator commonly used in fractional calculus. This operator is a notation for taking both the fractional derivative and the fractional integral into a single expression and is defined by

$$D_t^q = \begin{cases} \frac{d^q}{dt^q} & q > 0 \\ 1 & q = 0 \\ \int_0^t (d\tau)^{-q} & q < 0 \end{cases} . \tag{2.30}$$

There are several definitions for fractional derivatives [94]. The commonly used ones are the Grunwald-Letnikov, Riemann-Liouville, and Caputo definitions. The Grunwald-Letnikov definition is given by

$$
\begin{aligned}
D_t^q &= \frac{d^q f(t)}{d\,(t-a)^q} \\
&= \lim_{N\to\infty} \left[\frac{t-a}{N}\right]^{-q} \sum_{j=0}^{N-1} (-1)^j \binom{q}{r} f\left(t - j\left[\frac{t-a}{N}\right]\right). \qquad (2.31)
\end{aligned}
$$

The Riemann-Liouville definition is the simplest and easiest definition to use. It is given by

$$
\begin{aligned}
D_t^q &= \frac{d^q f(t)}{d\,(t-a)^q} \\
&= \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-q-1} f(\tau)\, d\tau, \qquad (2.32)
\end{aligned}
$$

where n is the first integer which is not less than q, i.e., $n-1 \leqslant q < n$ and Γ is the Gamma function,

$$
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \qquad (2.33)
$$

For functions $f(t)$ having n continuous derivatives for $t \geqslant 0$, where $n-1 \leqslant q < n$, the Grunwald-Letnikov and the Riemann-Liouville definitions are equivalent. The Laplace transforms of the Riemann-Liouville fractional integral and derivative are given as follows,

$$
L\{D_t^q f(t))\} = s^q F(s) \quad q \leqslant 0, \qquad (2.34)
$$

$$
\begin{aligned}
L\{D_t^q f(t))\} &= s^q F(s) - \sum_{k=0}^{n-1} s^k D_t^{q-k-1} f(0) \\
& \qquad n-1 < q \leqslant n \in N. \qquad (2.35)
\end{aligned}
$$

Unfortunately, the Riemann-Liouville fractional derivative appears unsuitable to be treated by the Laplace transform technique, since it requires the knowledge of the non-integer order derivatives of the function at $t = 0$. This problem does not exist in the Caputo definition that is sometimes referred to as smooth fractional derivative in literature. This definition of derivative is given as

$$
D_t^q f(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}} d\tau, & m-1 < q < m \\ \frac{d^m}{dt^m} f(t), & q = m \end{cases}, \qquad (2.36)
$$

where m is the first integer larger than q. It is found that the equations with Riemann-Liouville operators are equivalent to those with Caputo operators under the homogeneous initial conditions assumption [94]. The Laplace transform of the Caputo fractional derivative is

$$L\left\{D_t^q f(t)\right\} = s^q F(s) - \sum_{k=0}^{n-1} s^{q-1-k} f^{(k)}(0), \quad n-1 < q \leqslant n \in N. \tag{2.37}$$

Contrary to the Laplace transform of the Riemann-Liouville fractional derivative, only integer order derivatives of the function f appear in the Laplace transform of the Caputo fractional derivative. For zero initial conditions, Eq. (2.37) reduces to

$$L\left\{D_t^q f(t)\right\} = s^q F(s). \tag{2.38}$$

In this thesis, the notation D_t^q indicates the Caputo fractional derivative.

2.7.4 Stability of fractional-order systems

Starting from Eq. (2.30), it is possible to study the stability of fractional-order systems. A fractional-order differential equation with $0 < \alpha < 1$ typically presents a stability region that is larger than that of the same equation with integer order $\alpha = 1$ [2]. Consider a system with an n-dimensional state vector $x(t)$ taking values over $\mathbb{R}^n$ (i.e., all state variables are real) and evolving with the time variable t. All the results in chapter 3 are ultimately based on the following "α-stability" lemma.

Lemma 2.4 *[80]. Consider the linear fractional-order system*

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t), \quad \text{with} \quad x(0) = x_0, \tag{2.39}$$

where $0 < \alpha < 1$*,* $x \in \mathbb{R}^n$*, and* $A \in \mathbb{R}^{n\times n}$ *is a constant matrix with eigenvalues* $\xi_1, \ldots, \xi_n$*. System (2.39) is asymptotically stable around* 0 *if, and only if,*

$$|\arg(\xi_i)| > \frac{\alpha\pi}{2}, \quad i = 1, \ldots, n. \tag{2.40}$$

Throughout this thesis, "α-stable matrix A" means that all eigenvalues of matrix A satisfy condition (2.40) and, as a consequence, $\lim_{t\to\infty} x(t) = 0$ for all $x_0 \in \mathbb{R}^n$. If Eq. (2.40) holds with $\alpha = 1$, then the matrix A is Hurwitz (and $\lim_{t\to\infty} x(t) = 0$ as well).

2.7.5 Simulation of fractional-order systems

The numerical simulation of a fractional differential equation is as not simple as that of an ordinary differential equation. In the field of fractional chaos, two approximation methods have been proposed for the numerical integration of a fractional differential equation. The first method is based on the approximation of the fractional-order system behavior in the frequency domain. To simulate a fractional order system by using the frequency domain approximations, the fractional order equation of the system is first considered in the frequency domain and then the Laplace transform of the fractional integral operator is replaced by its integer order approximation. Then the approximate equations in frequency domain are transformed back into the time domain. The resulting ordinary differential equations can be numerically solved by applying well-known numerical methods. However, it has been shown that the simulation results using this approach can be very far from the reality [106].

The other algorithm to find an approximation for fractional-order systems is based on the predictor-corrector scheme [36, 37]. This method is an improved version of Adams-Bashforth-Moulton algorithm [37, 38, 63]. Consider the following differential equation:

$$\begin{aligned} D_t^q y(t) &= r\left(t, y(t)\right)), \quad 0 \leqslant t \leqslant T, y^{(k)}(0) = y_0^{(k)}, \\ k &= 0, 1, \cdots, m-1. \end{aligned} \tag{2.41}$$

This differential equation is equivalent to the Volterra integral equation [39]

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} r\left(s, y\left(s\right)\right) ds \tag{2.42}$$

with $m-1 < q \leqslant m$. Now, set $h = T/N$, $t_n = nh$ $(n = 0, 1, \cdots, N)$. Then Eq. (2.42) can be discretized as follows,

$$\begin{aligned} y_h\left(t_{n+1}\right) &= \sum_{k=0}^{m-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^q}{\Gamma(q+2)} r\left(t_{n+1}, y_h^p\left(t_{n+1}\right)\right) \\ &+ \frac{h^q}{\Gamma\left(q+2\right)} \sum_{j=0}^{n} a_{j,n+1} r\left(t_j, y_h\left(t_j\right)\right), \end{aligned} \tag{2.43}$$

where the predicted value $y_h^p\left(t_{n+1}\right)$ is determined by

$$y_h^p\left(t_{n+1}\right) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} r\left(t_j, y_h\left(t_j\right)\right) \tag{2.44}$$

and

$$
\begin{aligned}
a_{j,n+1} &= \begin{cases} n^{q+1}-(n-q)(n+1)^{q}, & j=0 \\ (n-j+2)^{q+1}+(n-j)^{q+1}-2(n-j+1)^{q+1}, & 1 \leqslant j \leqslant n \ , \\ 1 & j=n+1 \end{cases} \\
b_{j,n+1} &= \frac{h^{q}}{q}\left((n+1-j)^{q}-(n-j)^{q}\right). \qquad (2.45)
\end{aligned}
$$

The estimation error of this approximation can be shown to be [37]

$$
\max_{j=0,1,\cdots,N} |y(t_j)-y_h(t_j)| = \mathrm{O}\left(h^{p}\right),
$$

where $p=\min(2,1+q)$.

A fractional-order system can be numerically integrated by applying the method described above. Consider the following fractional-order system:

$$
\begin{cases} \frac{d^{q_1}x}{dt^{q_1}} = f_1(x,y,z) \\ \frac{d^{q_2}y}{dt^{q_2}} = f_2(x,y,z) \\ \frac{d^{q_1}z}{dt^{q_3}} = f_3(x,y,z) \end{cases} ; \quad 0<q_i \leqslant, \quad i=1,2.3, \qquad (2.46)
$$

with initial condition (x_0, y_0, z_0). The above system can be discretized as follows

$$
\begin{cases}
x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)}\left[f_1\left(x_{n+1}^{p}, y_{n+1}^{p}, z_{n+1}^{p}\right) + \sum\limits_{j=0}^{n} \alpha_{1,j,n+1} f_1(x_j,y_j,z_j)\right] \\
y_{n+1} = y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)}\left[f_2\left(x_{n+1}^{p}, y_{n+1}^{p}, z_{n+1}^{p}\right) + \sum\limits_{j=0}^{n} \alpha_{2,j,n+1} f_2(x_j,y_j,z_j)\right] \\
z_{n+1} = z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)}\left[f_3\left(x_{n+1}^{p}, y_{n+1}^{p}, z_{n+1}^{p}\right) + \sum\limits_{j=0}^{n} \alpha_{3,j,n+1} f_3(x_j,y_j,z_j)\right]
\end{cases} \qquad (2.47)
$$

where

$$
\begin{cases}
x_{n+1}^{p} = x_0 + \frac{1}{\Gamma(q_1)} \sum\limits_{j=0}^{n} \beta_{1,j,n+1} f_1(x_j,y_j,z_j) \\
y_{n+1}^{p} = y_0 + \frac{1}{\Gamma(q_2)} \sum\limits_{j=0}^{n} \beta_{2,j,n+1} f_2(x_j,y_j,z_j) \\
z_{n+1}^{p} = z_0 + \frac{1}{\Gamma(q_3)} \sum\limits_{j=0}^{n} \beta_{3,j,n+1} f_3(x_j,y_j,z_j)
\end{cases} ,
$$

$$
\alpha_{i,j,n+1} = \begin{cases} n^{q_i+1}-(n-q)(n+1)^{q_i}, & j=0 \\ (n-j+2)^{q_i+1}+(n-j)^{q_i+1} \\ \quad -2(n-j+1)^{q_i+1}, & 1 \leqslant j \leqslant n \\ 1, & j=n+1 \end{cases} \qquad (2.48)
$$

and

$$\beta_{i,j,n+1} = \frac{h^{q_i}}{q_i} \left((n+1-j)^{q_i} - (n-j)^{q_i} \right).$$

We have simulated all the fractional-order models in the present work by means of this method.

2.8 Summary

We have introduced several concepts, mathematical definitions and theorems that are used in subsequent chapters. Since the synchronized networks investigated in this thesis have nodes with chaotic dynamics, the basic concepts of chaos have been reviewed in the first place. As the problem of synchronization reduces to a zero convergence problem, the main concepts and theorems on stability and, in particular, zero convergence of a dynamic system, are provided in this chapter. LMI's are powerful tools for analyzing the stability of dynamical systems, hence we have introduced this object and reviewed some of its main properties. We then introduced the concept of fractional-order systems, which is the governing dynamics of two synchronized networks investigated in this thesis. Chaos synchronization in its simple form, and the basic concepts of complex networks have been provided as the required preliminaries for the statement of network outer synchronization, that is the main problem in this thesis. We have also addressed some common assumptions on coupling matrix of the synchronized networks, which we aim to relax in our approach to the problem of the outer synchronization. Finally, we have reviewed the concept of fractional-order differential equations, that is central to the analysis of outer synchronization to be introduced in Chapter 3.

Chapter 3

Synchronization of fractional-order networks

3.1 Introduction

In recent years, the study of the dynamics of fractional-order differential systems [27, 58] has attracted the interest of many researchers. In particular, it has been shown that some fractional-order differential systems behave chaotically or hyperchaotically, such as the fractional-order Chua circuit [49], the fractional-order Chen system [75], the fractional-order Lu system [74], and others [8, 1]. Following these findings, the synchronization of chaotic fractional-order systems has become a popular research topic due to its potential applications in secure communications and control [80]. For example, in [35] the synchronization of two fractional-order Lu systems has been studied. Also, the synchronization of two perturbed fractional-order Chen systems and the synchronization of two fractional-order Chua systems have been investigated in [15] and [70], respectively. Some additional attempts to attain synchronization of fractional-order systems can be found in [73, 34, 46].

The study of synchronization phenomena in complex dynamical networks whose nodes are governed by fractional-order nonlinear differential equations has also been addressed recently. Although complex networks have been a mainstream area of research for over a decade [103, 85], nearly all the effort has been devoted to systems where the dynamics of the individual nodes are modeled by integer-order (albeit possibly nonlinear) differential equations. Results on the synchronization of complex dynamical networks of fractional-order nodes have only been reported recently [112, 105] and they are limited to specific network configurations (such as the star topology in [112] or the ring topology of [105]).

In this chapter, we first investigate outer synchronization between two networks with diffusively-coupled, fractional-order dynamical nodes and then extend the analysis to relax several assumptions on the coupling scheme of the networks. In particular, we first obtain a linearized version of the synchronization error dynamics and then carry out a stability analysis that provides simple sufficient and necessary conditions for the synchronization error to converge locally toward zero[1]. Our approach avoids the need to compute eigenvalues of large system matrices (only their relative position is relevant) or to impose restrictive assumptions on the structure of the coupling matrices of the networks. Although we first state our main results for the case of two identical networks with known parameters, we also show how they can be extended to systems in which the network parameters are perturbed and, therefore, they are neither identical nor exactly known. This extension is based on an alternative formulation of the conditions for the convergence of the synchronization error in terms of LMI's. Under some assumptions on the coupling matrices, we also provide analytical results regarding the generalized synchronization of the networks[2].

The rest of the this chapter is organized as follows. Section 3.2 is devoted to a formal description of the network model and a statement of the synchronization problem to be addressed. In Section 3.3 we introduce our main results on the synchronization of two identical complex networks with fractional-order dynamical nodes. The extension to perturbed networks is carried out in Section 3.4. Numerical examples are presented in Section 3.5 and, finally, Section 3.6 is devoted to a brief summary and discussion of the obtained results.

[1] Specifically, we find sufficient and necessary conditions for the fractional-order differential equations governing the dynamics of the synchronization error, $\overline{e}(t)$, to have a fixed point at $\overline{e}(t) = 0$.

[2] Two systems A and B are in a generalized synchronization status when the state of the system B can be obtained as a deterministic transformation of the state of the system A.

3.2 Network model

Consider a network that consists of N identical nodes, each one being an n-dimensional system of fractional-order differential equations given by

$$\frac{d^{\alpha}x_i(t)}{dt^{\alpha}} = f(x_i(t)) + \sum_{j=1}^{N} c_{ij}\chi x_j(t), \quad i = 1, 2, \ldots, N, \tag{3.1}$$

where $0 < \alpha < 2$ is the derivative degree, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous differentiable function that describes the dynamics of the individual nodes, $x_i(t) \in \mathbb{R}^n$ is the state vector of node i at time t, $\chi \in \mathbb{R}^{n\times n}$ is a constant matrix with 0-1 elements linking coupled scalar variables and the matrix $C = (c_{ij}) \in \mathbb{R}^{N\times N}$ indicates the coupling configuration among the nodes of the network. Specifically, $c_{ij} > 0$ when there is a link from node j to node i $(i \neq j)$ and $c_{ij} = 0$ otherwise. If positive, the entry c_{ij} indicates also the strength of the connection between the nodes i and j.

As mentioned earlier, matrix C is of great importance for the behavior of the network. In the literature, a number of assumptions are commonly made in order to simplify the analysis of the class of systems described by Eq. (3.1). The most common assumed properties are listed in Section 2.6.3.

In this chapter, we show that these assumptions can often be relaxed. Indeed, we prove in chapter, that appropriate synchronization schemes can be found without assuming diffusivity, symmetry, balance or irreducibility.

Before definition of the response network, we introduce the open-plus-closed-loop (OPCL) scheme. Since the pioneering work in [55], the concept of OPCL has aroused new interest in nonlinear control problems. Particularly, it has been used to achieve outer synchronization between identical complex networks (governed by ordinary differential equations) [69, 64]. This technique is very general and has advantages of both open loop and closed loop control schemes. The original application of the OPCL technique was on integer-order systems, hence the matrix H was assumed to be Hurwitz. We will show that, for our purpose, it is enough that matrix H be α-stable to achieve synchronization. The Hurwitz condition for integer-order systems so becomes a special case. We refer the reader to [65, 48] and references therein for some other applications of the OPCL method in synchronization.

In order to investigate outer synchronization between two identical networks we consider Eq. (3.1) as the master network and assume there is a response system coupled with the master network in an OPCL scheme

[55], namely

$$\begin{aligned}\frac{d^{\alpha}y_i\left(t\right)}{dt^{\alpha}} &= f\left(y_i\left(t\right)\right)+\left(H-\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i(t)}\right)\left(y_i\left(t\right)-x_i\left(t\right)\right)\\ &\quad +\sum_{j=1}^{N}c_{ij}\chi y_j\left(t\right), \quad i=1,2,\cdots,N \end{aligned} \tag{3.2}$$

where $y_i\left(t\right)\in\mathbb{R}^n$ is the state vector of node i in the response system at time t, while α, N, f, c_{ij} and χ are the same as an Eq. (3.1). The second term in the right side of Eq. (3.2) is the synchronizer signal which is obtained using the OPCL method. In particular, H is a constant matrix and $\frac{\partial f}{\partial x}$ denotes the Jacobian matrix of function $f:\mathbb{R}^n\rightarrow\mathbb{R}^n$, which is evaluated at the point $x=x_i(t)$ in Eq. (3.2). The master network of Eq. (3.1) and the response network of Eq. (3.2) synchronize when their state variables converge toward a common value, i.e., when $\lim\limits_{t\rightarrow\infty}y_i(t)-x_i(t)=0$ for every node i.

In the sequel, we assume that the matrix χ in Eqs. (3.1) and (3.2) is an $n\times n$ identity matrix, $\chi=I_n$. This is done for the sake of clarity, but the analysis can be extended for other values of χ.

3.3 Synchronization analysis

In this section, we investigate the phenomenon of outer synchronization between the coupled networks defined by Eqs. (3.1) and (3.2). We first obtain a linearized fractional-order model for the synchronization error in Section 3.3.1. Then, we introduce our main results in Section 3.3.2, followed by extensions using a formulation based on LMI's in Section 3.3.3 and considering generalized synchronization in Section 3.3.4. Finally, we provide a theorem, in Section 3.3.5, that relates the outer synchronization of the networks directly with the eigenvalues of C and H.

3.3.1 Error dynamics

Let us introduce the error signal $\overline{e}_i(t) = y_i(t) - x_i(t)$, whose fractional derivative yields

$$\begin{aligned}
\frac{d^\alpha \overline{e}_i(t)}{dt^\alpha} &= \frac{d^\alpha y_i(t)}{dt^\alpha} - \frac{d^\alpha x_i(t)}{dt^\alpha} \\
&= f(x_i(t) + \overline{e}_i(t)) - f(x_i(t)) + \left(H - \left.\frac{\partial f}{\partial x}\right|_{x=x_i(t)} \right) \overline{e}_i(t) \\
&\quad + \sum_{j=1}^{N} c_{ij} \overline{e}_j(t) = G_{x_i(t)}(\overline{e}_i(t))
\end{aligned} \tag{3.3}$$

for $i = 1, ..., N$. When the errors vanish, i.e., $\lim_{t\to\infty} \overline{e}_i(t) = 0$ for $i = 1, ..., N$, the master and response networks given by Eqs. (3.1) and (3.2), respectively, synchronize. Unfortunately, it is hard to study the global stability of Eq. (3.3) around $\overline{e}_i(t) = 0$ because both the function f and its Jacobian $\frac{\partial f}{\partial x}$ are possibly nonlinear. To circumvent this difficulty, we propose to work with a linear approximation of the function $G_{x_i(t)}$ on the right-hand side of Eq. (3.3). Assuming the nonlinearity f is continuous and differentiable, a first-order Taylor series expansion of $G_{x_i(t)}(\overline{e}_i(t))$ around $\overline{e}_i(t) = 0$ yields

$$\begin{aligned}
G_{x_i(t)}(\overline{e}_i(t)) &\approx H\overline{e}_i(t) + \sum_{j=1}^{N} c_{ij}\overline{e}_j(t) \\
&= \hat{G}_{x_i(t)}(\overline{e}_i(t)),
\end{aligned} \tag{3.4}$$

where $\hat{G}_{x_i(t)}(\overline{e}_i(t))$ is a linear approximation of the fractional derivative of the error at time t. In the sequel, we adopt this approximation and study the stability of the set of equations

$$\begin{aligned}
\frac{d^\alpha e_i(t)}{dt^\alpha} &= \hat{G}_{x_i(t)}(e_i(t)) \\
&= He_i(t) + \sum_{j=1}^{N} c_{ij} e_j(t), \quad i = 1, \ldots, N.
\end{aligned} \tag{3.5}$$

Note that we change the notation and use $e_i(t)$ in order to explicitly indicate that this error signal is only an approximation of the true error $\overline{e}_i(t)$. The global stability of Eq. (3.5) around $e_i(t) = 0$ implies the *local* stability of the original Eq. (3.3) around $\overline{e}_i(t) = 0$. In practice, this means that there exists $\epsilon > 0$ such that the conditions

$$\lim_{t\to\infty} e_i(t) = 0, \quad \forall e_i(0), \quad \text{and} \|\overline{e}_i(0)\| < \epsilon \tag{3.6}$$

together imply that

$$\lim_{t\to\infty} \overline{e}_i(t) = 0.$$

Consequently, we adopt the following definition of *local* synchronization.

Definition 3.1 *The master network (3.1) and the response network (3.2) synchronize locally when* $\lim_{t\to\infty} e_i(t) = 0$*, irrespective of* $e_i(0)$*, for* $i = 1, ..., N$.

Several propositions in this paper have the form "the networks (3.1) and (3.2) synchronize locally if, and only if, the set of conditions $\mathcal{S}$ is satisfied". If the latter claim is true, then the alternative statement "the system of equations of the form of (3.3), with $i = 1, ..., N$, has a fixed point at $\overline{e}_i(t) = 0$ ($i = 1, ..., N$) if, and only if, the set of conditions $\mathcal{S}$ is satisfied" is also true.

Some attempts to extend classical nonlinear control techniques based on Lyapunov functions for fractional-order systems can be found in [67, 68, 108]. However, there is still a gap between the theoretical results in those papers and practical applications, hence they have not enjoyed much use in real-world control problems so far. For this reason, we do not pursue a direct analysis of the nonlinear Eq. (3.3) but rely on the linearization error of Eq. (3.5) and the notion of α-stability in Lemma 2.4 for our analysis. This approach to the analysis of the stability of fractional-order differential equations has been already followed by other authors [132, 107].

All the results in this paper are obtained by way of Lemma 3.1 below, whose statement requires the introduction of some additional notations. Consider an $n \times n$ matrix A and an $m \times m$ matrix B with eigenvalues and eigenvectors $(x_i, \nu_i) \in \mathbb{R}^n \times \mathbb{R}$ and $(y_i, \mu_i) \in \mathbb{R}^m \times \mathbb{R}$, respectively, i.e.,

$$\begin{aligned} Ax_i &= \nu_i x_i, \quad i = 1, \ldots, n, \quad \text{and} \\ By_i &= \mu_i y_i, \quad j = 1, \ldots, m, \end{aligned} \tag{3.7}$$

where $x_i = [x_{i,1}, x_{i,2}, \ldots, x_{i,n}]^\top$ and $y_j = [y_{j,1}, y_{j,2}, \ldots, y_{j,m}]^\top$ ($^\top$ denotes transposition). Let us also introduce the $nm \times nm$ matrix T as

$$T = A \otimes I_m + I_n \otimes B \tag{3.8}$$

where $\otimes$ denotes the Kronecker product. The eigenvalues and eigenvectors of T are denoted as τ_i and z_i , respectively, with $i = 1, \ldots, nm$, i.e.,

$$Tz_i = \tau_i z_i, \tag{3.9}$$

where $z_i = [z_{i,1}, z_{i,2}, \ldots, z_{m,n}]^\top$.

The following lemma makes a connection between the eigenvalues and eigenvectors of T and those of A and B.

Lemma 3.1 *Let $X = [x_1, \ldots, x_n] \in \mathbb{R}^{n\times n}$ and $Y = [y_1, \ldots, y_m] \in \mathbb{R}^{m\times m}$ be the matrices whose columns are the eigenvectors of A and B, respectively. The eigenvectors of T have the form $z_i = \psi_i (X \otimes Y)$, $i = 1, \cdots, nm$, where $\psi_i (M)$ is the operator that selects the i-th column of matrix M. The eigenvalues of T have the form*

$$\tau_i = \nu_k + \mu_j, \tag{3.10}$$

where $i = 1, 2, \ldots, mn$, $k = \lfloor i/n \rfloor + 1$ and $j = i + n - kn$. (For a real number $r \in \mathbb{R}$, $\lfloor r \rfloor$ is the floor operator, i.e., $\lfloor r \rfloor = max \{a \in \mathbb{Z} : a \leq r\}$.)

Proof. Let $t_{i,j}$ denote the element in the i-th row and j-th column of matrix T and consider the indices $w = 0, \ldots, n-1$ and $c = 1, \ldots, m$. If we define $l = wm + c$, it is apparent that l runs from 1 to nm and it can be used as an index for the rows of T. In particular, the entries of the l-th row of T have the form

$$\left.\begin{array}{rcll} t_{l,wm+j} & = & b_{c,j}, & \text{for } j = 1, \ldots, m, \quad j \neq c \\ t_{l,l} & = & b_{c,c} + a_{(w+1),(w+1)} & \\ t_{l,pm+c} & = & a_{w+1,p}, & \text{for } p = 0, \ldots, n-1, \quad p \neq w \\ 0, & & & \text{otherwise} \end{array}\right\}. \tag{3.11}$$

Consider now two arbitrary eigenpairs of A and B, (x_i, ν_i) and (y_j, μ_j), respectively, for some $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. We only need to show that $\tau = \nu_i + \mu_j$ is an eigenvalue of matrix T with an $nm \times 1$ eigenvector z defined as

$$z = x_i \otimes y_j = \left[\underbrace{x_{i,1}y_{j,1}}_{z_1}, \underbrace{x_{i,1}y_{j,2}}_{z_2}, \ldots, \underbrace{x_{i,n}y_{j,m-1}}_{z_{n(m-1)}}, \underbrace{x_{i,n}y_{j,m}}_{z_{nm}} \right]^\top. \tag{3.12}$$

This is relatively straightforward, however. Indeed, if we calculate the l-th entry of the vector Tz, denoted $(Tz)_l$, we obtain

$$(Tz)_l = \sum_{p=1}^{mn} t_{l,p} z_p. \tag{3.13}$$

Substituting Eq. (3.11) into Eq.(3.13) yields

$$(Tz)_l = \sum_{k=1}^{m} b_{c,k} x_{i,w+1} y_{j,k} + \sum_{p=1}^{n} a_{w+1,p} x_{i,p} y_{j,c} \tag{3.14}$$

and extracting common factors $x_{i,w+1}$ and $y_{j,c}$, we arrive at

$$(Tz)_l = x_{i,w+1} \sum_{k=1}^{m} b_{c,k} y_{j,k} + y_{j,c} \sum_{p=1}^{n} a_{w+1,p} x_{i,p}.$$

Finally, we note that $\sum_{k=1}^{m} b_{c,k} y_{j,k}$ is the c-th entry of the vector By_j and, similarly, $\sum_{p=1}^{n} a_{w+1,p} x_{i,p}$ is the $(w+1)$-th element of the vector Ax_i. Since y_j and x_i are eigenvectors of B and A, respectively, this means that $\sum_{k=1}^{m} b_{c,k} y_{j,k} = \mu_j y_{j,c}$ and $\sum_{p=1}^{n} a_{w+1,p} x_{i,p} = \nu_i x_{i,w+1}$. As a consequence,

$$\begin{aligned}(Tz)_l &= \mu_j y_{j,c} x_{i,w+1} + \nu_i y_{j,c} x_{i,w+1} \\ &= (\mu_j + \nu_i)\, y_{j,c} x_{i,w+1} \\ &= \tau z_l.\end{aligned}$$

Since this argument is valid for any $l \in \{1, \ldots, nm\}$, it follows that $(z, \tau) \in \mathbb{R}^{nm} \times \mathbb{R}$ in an eigenpair of T. Moreover, we can obtain every eigenpair of T by enumerating the pairs (x_i, ν_i) and (y_j, μ_j) for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. $\square$

A shorter proof for Eq. (3.10) can be found in [18, Lemma 3.24].

Now we come back to the error dynamics of Eq. (3.5). Since every $e_i(t)$ is an n-dimensional vector, we can define an nN-dimensional global error vector

$$e(t) = \left[e_1^\top(t), e_2^\top(t), \ldots, e_N^\top(t)\right] \in \mathbb{R}^{nN} \tag{3.15}$$

and the resulting error dynamics can be compactly written as

$$\frac{d^\alpha e(t)}{dt^\alpha} = (I_N \otimes H + C \otimes I_n)\, e(t). \tag{3.16}$$

3.3.2 Basic results

We analyze the stability of the nN-dimensional system of Eq. (3.16) around 0 by studying the position of the eigenvalues of the system matrix, $I_N \otimes H + C \otimes I_n$. Direct calculation of the eigenvalues of such a large matrix is prohibitive in practice, but using Lemma 3.1 and some properties of the coupling matrix C we can develop useful stability criteria without determining the exact position of the eigenvalues.

We start with two auxiliary results concerning the eigenvalues of matrix C and a basic property of complex numbers.

Lemma 3.2 *All the eigenvalues of a diffusive matrix C have nonpositive real parts, and 0 is an eigenvalue of matrix C. Moreover, if C is irreducible, 0 is an eigenvalue with multiplicity one.*

Proof: See [121].

Now we are ready to introduce our main results regarding the synchronization of the networks (3.1) and (3.2). We initially assume the matrix C to be diffusive for simplicity.

Theorem 3.1 *Networks (3.1) and (3.2) with symmetric and diffusive coupling matrix C synchronize locally if, and only if, matrix H is α-stable, where $\alpha \in (0, 1]$.*

Proof: We prove sufficiency first. Assume H is α-stable and let $\lambda_1, \cdots, \lambda_N$ denote the eigenvalues of matrix C. Since this matrix is symmetric and real, $\lambda_i \in \mathbb{R}\ \forall i$ and, according to Lemma 3.2, we can sort them out in decreasing order as

$$0 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N, \tag{3.17}$$

i.e., they are all nonpositive.

Now, let $\xi_1, \cdots, \xi_N$ be the eigenvalues of H. Since H is α-stable, they all satisfy the stability condition of Eq. (2.40). Moreover, from Lemma 3.1, all eigenvalues of $I_N \otimes H + C \otimes I_n$ have the form $\lambda_i + \xi_j$ for some $i \in \{1, \cdots, N\}$ and $j \in \{1, \cdots, n\}$. Since ξ_j satisfies the stability condition of Eq.(2.40) and λ_i is real and nonpositive, $\lambda_i + \xi_j$ also satisfies Eq. (2.40) and, as a consequence, $I_N \otimes H + C \otimes I_n$ is α-stable. From (3.16), if $I_N \otimes H + C \otimes I_n$ is α-stable then $\lim_{t\to\infty} e(t) = 0$.

Now we prove necessity by contradiction. Assume that H is not $\alpha-$stable. As a consequence, there exists some $j \in \{1, \cdots, n\}$ such that $|\arg(\xi_j)| \leq \frac{\alpha\pi}{2}$. Moreover, from Lemma 3.2, C has at least one null eigenvalue, i.e., $\exists i \in \{1, \cdots, N\}$ such that $\lambda_i = 0$. Therefore, $\tau_{i,j} = \lambda_i + \xi_j = \xi_j$ is an eigenvalue of $I_N \otimes H + C \otimes I_n$ such that $|\arg(\tau_j)| \leq \frac{\alpha\pi}{2}$ and, as a consequence, $I_N \otimes H + C \otimes I_n$ is not α-stable (hence the networks are not synchronized). $\square$

Theorem 3.2 *The integer-order networks (3.1) and (3.2) with diffusive coupling matrix C and $\alpha = 1$ synchronize locally if, and only if, the matrix H is Hurwitz.*

Proof: Assume H is Hurwitz and let $\lambda_1, \cdots, \lambda_N$ denote the eigenvalues of matrix C. From Lemma 3.2, we obtain that

$$\mathfrak{Real}(\lambda_i) \leq 0, \quad \forall i. \tag{3.18}$$

Now, let $\xi_1, \cdots, \xi_N$ be the eigenvalues of H. Since H is Hurwitz, they all have negative real parts, i.e.,

$$\mathfrak{Real}(\xi_i) < 0, \quad \forall i. \tag{3.19}$$

Moreover, from Lemma 3.1, all eigenvalues of $I_N \otimes H + C \otimes I_n$ have the form $\lambda_i + \xi_j$ for some $i \in \{1, \cdots, N\}$ and $j \in \{1, \cdots, n\}$. From Eq. (3.18) and (3.19), it is easy to find that all these eigenvalues have negative real parts. i.e.,

$$\mathfrak{Real}(\xi_i + \lambda_j) < 0, \quad \forall i, j. \tag{3.20}$$

which means that the system matrix in Eq. (3.16) is Hurwitz and, hence, $\lim_{t\to\infty} e(t) = 0$.

We prove necessity by contradiction, in a way similar to the proof of Theorem 3.1. Assume that H is not Hurwitz. As a consequence, there exists some $j \in \{1, \cdots, n\}$ such that $\mathfrak{Real}(\xi_j) > 0$. Moreover, from Lemma 3.2, C has at least one null eigenvalue, i.e., $\exists i \in \{1, \cdots, N\}$ such that $\lambda_i = 0$. Therefore, $\tau_{i,j} = \lambda_i + \xi_j = \xi_j$ is an eigenvalue of $I_N \otimes H + C \otimes I_n$ such that $\mathfrak{Real}(\tau_j) > 0$ and, as a consequence, $I_N \otimes H + C \otimes I_n$ is not Hurwitz, which implies that the networks do not synchronize. □

Let us note that the same criterion (H being Hurwitz) was stated in [69], but it was given as a sufficient condition only. This was a consequence of using the Lyapunov stability theorem, which yields only a sufficient condition for stability.

If we remove the symmetry assumption in Theorem 3.1, we can still provide a sufficient condition for outer synchronization.

Theorem 3.3 *Networks (3.1) and (3.2) with diffusive coupling matrix C synchronize locally if matrix H is Hurwitz, for any $\alpha \in (0, 1]$.*

Proof: The proof follows the same argument as the first part of the proof of Theorem 3.2, hence it is omitted here. □

It is important to notice that once we have designed a synchronization scheme (i.e., the matrix H) for a system with degree $\alpha \leq 1$, then the same scheme is valid for any system of a lower fractional-order degree, as shown by the following corollary.

Corollary 3.1 *If matrix H synchronizes locally the networks (3.1) and (3.2) with derivative degree $\alpha_1 \leq 1$ and symmetric diffusive coupling matrix C, then H also synchronizes locally the same networks with any derivative degree α_2 such that $\alpha_2 < \alpha_1$. In particular, if H synchronizes locally the networks (3.1) and (3.2) with integer-order derivatives, then it also works as a local synchronizer for the same networks with fractional degree α, $0 < \alpha < 1$.*

Proof: This is a straightforward consequence of Theorem 3.1. $\square$

3.3.3 Synchronization analysis based on LMI's

We can also analyze the synchronization of the master and response networks based on a different set of conditions defined by systems of LMI's, which is previously defined in Definition 2.6 in Section 2.3 of Chapter 2.

We first provide a criterion for the convergence of the synchronization error, $\lim_{t\to\infty} e(t) = 0$, similar to Theorem 3.1, but stated in terms of suitable LMI's.

Theorem 3.4 *The networks (3.1) and (3.2) with diffusive coupling matrix C synchronize locally if, and only if, there exist positive definite matrices Z_1 and Z_2 that satisfy the following LMI's (where $r = \exp\left\{j(1-\alpha)\frac{\pi}{2}\right\}$, $\bar{r}$ denotes its conjugate and $j = \sqrt{-1}$):*

(i) For $0 < \alpha < 1$ and C symmetric,

$$\bar{r}Z_1H^T + rHZ_1 + rZ_2H^T + \bar{r}HZ_2 < 0. \tag{3.21}$$

(ii) For $\alpha = 1$,

$$Z_1H + H^TZ_1 + Z_2 < 0. \tag{3.22}$$

(iii) For $1 < \alpha < 2$ and C symmetric,

$$\begin{pmatrix} \left(H^TZ_1 + Z_1H\right)\sin\left(\alpha\frac{\pi}{2}\right) & \left(H^TZ_1 - Z_1H\right)\cos\left(\alpha\frac{\pi}{2}\right) \\ \left(Z_1H - H^TZ_1\right)\cos\left(\alpha\frac{\pi}{2}\right) & \left(H^TZ_1 + Z_1H\right)\sin\left(\alpha\frac{\pi}{2}\right) \end{pmatrix} + Z_2 < 0. \tag{3.23}$$

Proof. From [99, Thoerem 12], the LMI in (i) holds true for some positive definite matrices Z_1 and Z_2 if, and only if, the matrix H is α-stable. Therefore, from Theorem 3.1, the synchronization error, $e(t)$, converges to zero.

The LMI in (ii) holds for some positive definite Z_1 and Z_2 if, and only if, the matrix H is Hurwitz [77]. Then we simply apply Corollary 3.2 to obtain that the synchronization error $e(t)$ converges to zero also in this case.

In [3], it is shown that the LMI of (iii) holds true for some positive definite Z_1 and Z_2 if, and only if, matrix H is α-stable when $1 < \alpha < 2$. Then, similarly to the case (ii), we only need to apply Theorem 3.1 to show that the networks are synchronized. $\Box$.

Let us remark that the matrices Z_1 and Z_2 are not necessarily the same for the three cases. E.g., when $\alpha \in (0, 1)$ (and C is symmetric), we only need to find Z_1 and Z_2 such that Eq. (3.21) holds, without regard to Eqs. (3.22) and (3.23).

3.3.4 Generalized synchronization

Somewhat contrary to intuition, it is possible to attain synchronization between the master (3.1) and response (3.2) networks when the matrix H in (3.2) is nulled out, i.e., it is an $n \times n$ zero matrix, $H = 0_{n\times n}$. This is achieved, however, at the expense of imposing slightly different assumptions on the coupling matrix C.

In particular, let $\mathcal{C}(\varepsilon)$ be the set of $N \times N$ matrices with real entries such that the sum of the entries in each row is equal to the real number ε and the off-diagonal elements are nonnegative, i.e.,

$$\mathcal{C}(\varepsilon) \triangleq \left\{ (c_{i,j}) \in \mathbb{R}^{N\times N} : \sum_{j=1}^{N} c_{i,j} = \varepsilon, \quad \forall i, \text{ and } c_{i,j} \geq 0, \quad \forall i \neq j \right\}.$$

Theorem 3.5 *The networks (3.1) and (3.2) with derivatives of degree $0 < \alpha \leq 1$, coupling matrix $C \in \mathcal{C}(\varepsilon)$, $\varepsilon < 0$, and $H = 0_{n\times n}$ synchronize locally.*

Proof: The proof is straightforward from Lemma 3.1 in this paper and [121, Lemma 2]. Since all eigenvalues of matrix H are zero, Lemma 3.1 ensures that the eigenvalues of matrix $I_N \otimes H + C \otimes I_n$ are the same as those of matrix C, but with multiplicity n. From [121, Lemma 2], we know that all eigenvalues of matrix C are numbers with negative real parts. Therefore, all the eigenvalues of the system matrix in Eq. (3.16) also have negative real parts. Hence, they comply with Eq. (2.40) and the system matrix is either α-stable (for $\alpha < 1$) or Hurwitz (for $\alpha = 1$). $\Box$

The OPCL scheme with $H = 0_{n\times n}$ can also lead to the generalized synchronization [98] of the two networks. Let us adopt the following definition.

Definition 3.2 *Two dynamical systems with state vectors $x(t)$ and $y(t)$ are in generalized synchronization if there exists a function ϕ such that*

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \phi\left(x(t)\right).$$

This is a weaker form of synchronization that has received a good deal of attention [98, 127] because it allows to describe a broad class of phenomena that occur in network systems.

Here, we carry out an approximate analysis based on the linearized error of Eq. (3.5). In particular, we show that, provided $H = 0_{n\times n}$ and the coupling matrix C is suitable, the approximate error $e(t)$ converges to a constant value, $\lim_{t\to\infty} e(t) = e_\infty \neq 0$, which is a form of generalized (outer) synchronization.

Theorem 3.6 *The networks (3.1) and (3.2) with diffusive and irreducible coupling matrix C and $H = 0_{n\times n}$ are in a generalized synchronization state for the approximate error $e(t)$.*

Proof. The matrix C has a zero eigenvalue with multiplicity one. Since $H = 0_{n\times n}$, we can take columns of I_n as its eigenvectors. Hence, the eigenvectors associated with the zero eigenvalues of the system matrix in Eq. (3.16) are independent, which implies that the associated blocks of zero eigenvalues in Jordan canonical form are simple. Therefore, the synchronization errors may not necessarily converge to zero, but they are guaranteed to converge to fixed points, denoted η_i, $i = 1, \cdots, nN$. Thus, the coupled networks attain generalized synchronization, according to Definition 3.2, with the functions $y_{i,j}(t) = \phi_k\left(x_i(t)\right) = x_i(t) - \eta_k$, where $i = 1, \ldots, N$, $j = 1, \ldots, n$ and $k = (i-1)\,n + j$. $\square$.

Let us remark that Theorem 3.6 does not guarantee that the fractional-order differential Eq. (3.3), that describes the dynamics of the true error, $\overline{e}(t)$, has a fixed point at some $e_\infty \neq 0$. It only shows that the evolution of the approximate error $e(t)$ *suggests* that generalized outer synchronization can be achieved by the networks. In Section 3.5.1, we present a simple illustrative example that shows how generalized synchronization is actually attained.

3.3.5 Generic coupling matrix.

We can drop the assumptions of the coupling matrix being diffusive and irreducible, provided that we state a joint assumption on the eigenvalues of C and H.

Theorem 3.7 *Let $\lambda_1, \ldots, \lambda_N$ and $\xi_1, \ldots, \xi_n$ be the eigenvalues of C and H, respectively. Then, the networks (3.1) and (3.2) with coupling matrix C synchronize locally, for any $0 < \alpha \leq 1$, if*

$$\max_i (\mathfrak{Real}(\lambda_i)) + \max_j (\mathfrak{Real}(\xi_j)) < 0, \tag{3.24}$$

for $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, n\}$.

Proof. From (3.24) and Lemma 3.1, it can be found that all the eigenvalues of the system matrix in Eq. (3.16) are negative real numbers. Therefore, the system (3.16) satisfies the stability condition (2.40) for any $0 < \alpha \leq 1$ and, as a consequence, $\lim_{t\to\infty} e(t) = 0$. □

3.4 Robust synchronization

The goal of this section is to obtain necessary and sufficient conditions for "robust" synchronization, i.e., conditions that guarantee local synchronization even in cases of uncertainty in the parameter values, which can, therefore, be different in the master and the response networks. We first revisit the system model of Section 3.2 and rewrite it in a way that turns out more useful for the subsequent analysis. Then we provide sufficient and necessary conditions, in the form of a set of LMI's, for local synchronization with mismatched parameters.

3.4.1 Network model revisited

The dynamics of the nodes, both in the master (3.1) and response (3.2) networks, depend on the nonlinear function f. Let us explicitly write f in terms of its linear and nonlinear parts as

$$f(x) = Ax + F(x), \tag{3.25}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is a constant matrix and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinearity. Using Eq. (3.25) and following the same argument as in Section 3.3.1, the linearized dynamics of the synchronization error can be rewritten as

$$\frac{d^\alpha e_i(t)}{dt^\alpha} = (A + H)\, e_i(t) + \sum_{j=1}^{N} c_{i,j} e_j(t), \quad i = 1, \ldots, N. \tag{3.26}$$

Note that this is not a modification of either the system or its error dynamics, but simply an alternative way to represent them. In particular, all the results in Section 3.3 can be rewritten with this new formulation if we simply replace H by $A + H$ in Theorems 3.1 through 3.7.

3.4.2 Synchronization

We can extend Theorem 3.4 to systems that present some uncertainty in the available knowledge of the matrices A and H. To be specific, let us assume that A and H are perturbed as

$$A = A_n + \Delta A, \tag{3.27}$$
$$H = H_n + \Delta H, \tag{3.28}$$

where H_n and A_n denote the nominal values of the matrices and ΔH and ΔA are unknown, albeit bounded[3], perturbations. We handle both perturbations together by introducing the matrix

$$P = H + A = P_n + \Delta P, \tag{3.29}$$

where $P_n = H_n + A_n$ and $\Delta P = \Delta H + \Delta A$. Note that, since ΔH and ΔA are bounded, ΔP is also bounded and the perturbed matrix P can only take values in a convex set that we denote as P^I.

The following theorem provides necessary and sufficient conditions for the outer (local) synchronization of the networks that can be checked when only the nominal values, A_n and H_n, and the bounds for the perturbations ΔA and ΔH are known.

Theorem 3.8 *Assume the coupling matrix C is diffusive. The synchronization error of Eq. (3.26) converges to 0 if, and only if, there exists symmetric and positive definite matrices Z_1 and Z_2 such that, for all vertex matrices $P^* \in P^I$ and $r = \exp\left\{j(1-\alpha)\frac{\pi}{2}\right\}$, the following LMI's are satisfied:*

- *For $0 < \alpha < 1$ and C symmetric,*

$$\bar{r} Z_1 P^{*\top} + r P^* Z_1 + r Z_2 P^{*\top} + \bar{r} P^* Z_2 < 0. \tag{3.30}$$

- *For $\alpha = 1$,*

$$Z_1 P^* + P^{*\top} Z_1 + Z_2 < 0.$$

- *For $1 < \alpha < 2$ and C symmetric,*

$$\begin{pmatrix} \left(P^{*\top} Z_1 + Z_1 P^*\right) \sin\left(\alpha\frac{\pi}{2}\right) & \left(P^{*\top} Z_1 - Z_1 P^*\right) \cos\left(\alpha\frac{\pi}{2}\right) \\ \left(Z_1 P^{*\top} - P^* Z_1\right) \cos\left(\alpha\frac{\pi}{2}\right) & \left(P^{*\top} Z_1 + Z_1 P^*\right) \sin\left(\alpha\frac{\pi}{2}\right) \end{pmatrix} + Z_2 < 0.$$

[3] For a matrix $M \in \mathbb{R}^{r\times k}$, with entries $m_{i,j}$, $i = 1, ..., r$, $j = 1, ..., k$, we say that M is bounded if every entry is bounded, i.e., there exists a constant c such that $|m_{i,j}| < c$ for all i and j.

Proof: Following exactly the same arguments as in the proof of Theorem 3.4, we can prove that, when $P = P^*$ is a vertex of P^I, $\lim_{t\to\infty} e_i(t) = 0$ for all i if, and only if,

- the LMI of (i) holds, C is symmetric and $0 < \alpha < 1$, or
- the LMI of (ii) holds and $\alpha = 1$, or
- the LMI of (iii) holds, C is symmetric and $1 < \alpha < 2$.

However, if the LMI's hold for all vertex matrices P^*, then they also hold for every $P \in P^I$ (because both P^I and the sets defined by the LMI's are convex), hence $\lim_{t\to\infty} e_i(t) = 0$ for all i and all $P \in P^I$. $\square$

3.5 Numerical simulations

In this section we present some computer simulation results that illustrate the achievement of outer synchronization between two networks in some of the scenarios considered in Sections 3.3 and 3.4. These simulations are relevant because we can numerically integrate the true synchronization errors $\bar{e}_i(t) = y_i(t) - x_i(t)$, $i = 1, ..., N$, governed by the differential Eq. (3.3), while the analysis of the previous sections is based on the approximate errors $e_i(t)$, obtained by a linearization of the right-hand-side of Eq. (3.3), whose dynamics is governed by Eq. (3.5).

3.5.1 Generalized synchronization

We first illustrate how generalized outer synchronization can be achieved between two coupled networks using $H = 0$ in the OPCL scheme of Eq. (3.2). With that aim, we consider two diffusively-coupled networks with ten nodes each ($N = 10$). The general structure of the system abides by Eqs. (3.1) and (3.2), i.e., we have a master network and a response network, but the differential equations governing the dynamics of the nodes are of integer order ($\alpha = 1$). The nonlinear function f corresponds to the Lorenz system, that is,

$$f(x_i(t)) = \begin{pmatrix} -\sigma(x_{i,1}(t) - x_{i,2}(t)) \\ \rho x_{i,1}(t) - x_{i,2}(t) - x_{i,1}(t)x_{i,3}(t) \\ \beta x_{i,3}(t) + x_{i,1}(t)x_{i,2}(t) \end{pmatrix}, \qquad (3.31)$$

for $i = 1, ..., N$, where $\sigma = 10$, $\rho = 28$ and $\beta = 2/3$. The coupling matrix is

$$C = \begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -4 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -5 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -3 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & -5 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 & -5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -3 \end{pmatrix},$$

which can be shown to be diffusive and irreducible. A similar example was considered in [69], but in that paper the matrix H was chosen to be non-null in order to achieve identical synchronization.

This system matches the assumptions of Theorem 3.6, which establishes that the linearized error $e(t)$ converges to a (possibly non-zero) constant value when $H = 0_{3\times 3}$. We recall that $e(t)$ is only an approximation of the true error $\overline{e}(t)$, hence we need to check numerically whether the latter converges as suggested by the approximate analysis. This is indeed the case, as observed in Figure 3.1, that shows the convergence of the true synchronization errors, $\overline{e}_{i,j}(t) = x_{i,j}(t) - y_{i,j}(t)$, for $i = 1, ..., 10$, $j = 1, ..., 3$ and $0 \leq t \leq 5$. It can be seen that all errors converge to fixed, albeit possibly non zero, points. This is a simple case of generalized synchronization according to Definition 3.2.

3.5.2 Robust synchronization

Unlike the simulation of an integer-order differential equation, the numerical simulation of a fractional differential equation is not straightforward. In order to obtain the results presented in this section, we have applied the predictor-corrector scheme of [36, 37], an improved version of the Adams-Bashforth-Moulton algorithm [37, 38, 63].

We now consider two diffusively coupled networks with $N = 23$ nodes each, scale-free structure and fractional-order Lorenz dynamics, with $\alpha = 0.95$. In order to illustrate with numerical results the analysis of Section 3.4, we (a) introduce bounded perturbations in the parameters of the node dynamical equations and (b) select a non-null synchronizer-matrix H in order to show how robust synchronization can be attained.

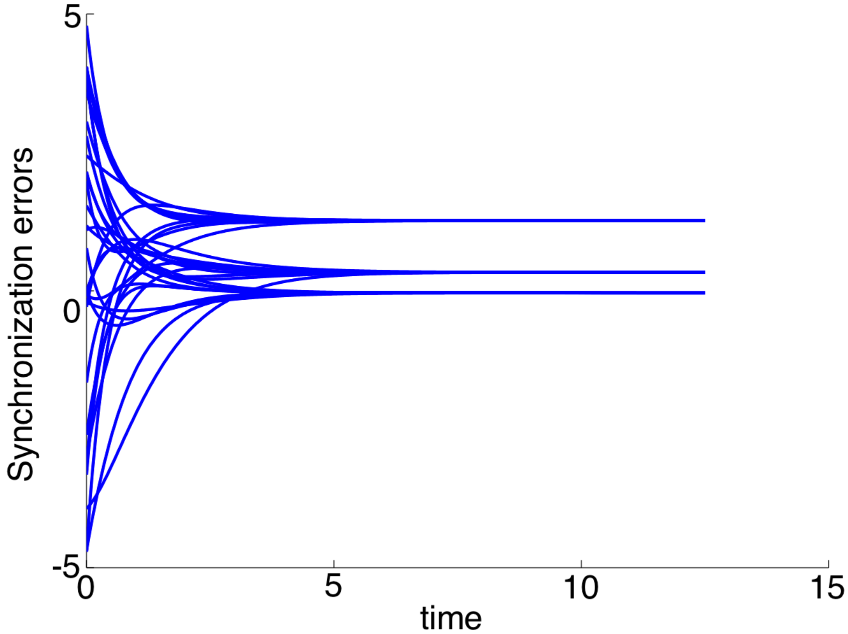

Figure 3.1: Generalized outer synchronization between two networks with Lorenz dynamics. There are $N = 10$ nodes in each network and the order of the differential equations is integer, $\alpha = 1$. All errors converge to fixed (albeit non-zero) points.

The dynamics of the nodes is determined by the same function f as in Eq. (3.31), which we decompose f into linear and nonlinear parts as

$$f\left(x_i(t)\right) = \overbrace{\begin{pmatrix} -\tilde{\sigma} & \tilde{\sigma} & 0 \\ \tilde{\rho} & -1 & 0 \\ 0 & 0 & \tilde{\beta} \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} x_{i,1}\left(t\right) \\ x_{i,2}\left(t\right) \\ x_{i,3}\left(t\right) \end{pmatrix}}^{x_i(t)} + \overbrace{\begin{pmatrix} 0 \\ -x_{i,1}(t)x_{i,3}(t) \\ x_{i,1}(t)x_{i,2}(t) \end{pmatrix}}^{F(x_i(t))}, \quad (3.32)$$

where $\tilde{\sigma} = \sigma + \Delta_1$, $\tilde{\rho} = \rho + \Delta_2$ and $\tilde{\beta} = \beta + \Delta_3$ are perturbed parameters, with nominal values $(\sigma, \rho, \beta) = (10, 28, 2/3)$ and bounded perturbations $|\Delta_l| \leq 1$, $l = 1, 2, 3$.

Th structure of the scale-free network used in this section is depicted in Figure 3.2. The 23×23 coupling matrix that determines the connectivity of the network is

$$
C = \left(\begin{array}{ccccccccccccccccccccccc}
-6-\gamma_1 & 1 & 2 & 1 & 1 & \gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -3-\gamma_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & -6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2-\gamma_3 & \gamma_3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2
\end{array}\right), \tag{3.33}
$$

where γ_l, $l = 1, 2, 3$ are unknown bounded perturbations ($|\gamma_l| \leq 1$ for all l). Note that, even if perturbations of C are not explicitly considered in Theorem 3.8, the matrix of (3.33) is diffusive independently of the exact values of $\gamma_{l=1,2,3}$ and hence we actually abide by the assumptions of Theorem 3.8. The synchronizer matrix is

$$
H = \overbrace{\begin{pmatrix} 0 & 0 & 0 \\ -20 & 10 & 0 \\ 0 & 0 & -4 \end{pmatrix}}^{H_n} + \overbrace{\begin{pmatrix} 0 & 0 & 0 \\ +\Delta_4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{\Delta H} \tag{3.34}
$$

where $-1 \leq \Delta_4 < 1$ is a bounded perturbation.

Similar to H, the matrix A can be split into its nominal value and a

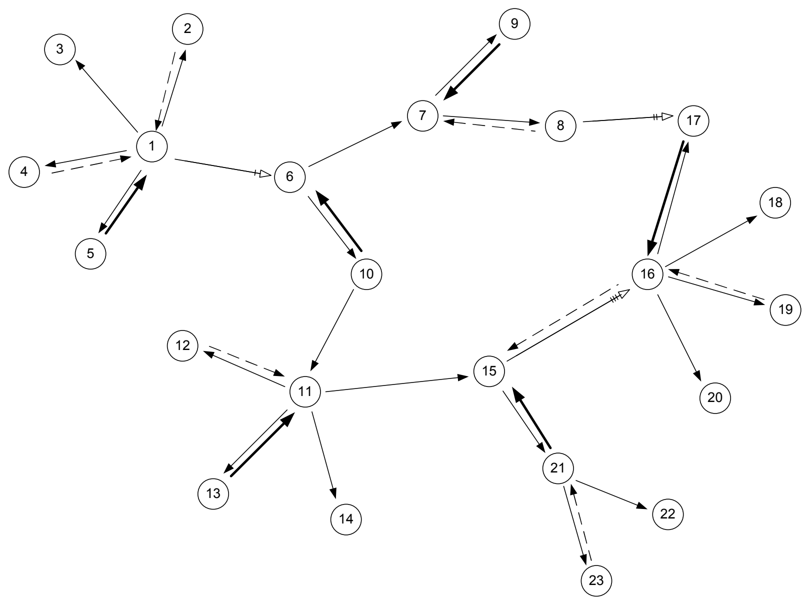

Figure 3.2: Graphical representation of the topology determined by the coupling matrix C of Eq. (3.33). The links represented with thin solid lines correspond to coefficients of the form $c_{ij} = 1$; those with dashed lines correspond to coefficients of the form $c_{ij} = 2$; and the links with thick solid lines correspond to coupling coefficients of the $c_{ij} = 3$. The "uncertain" links, of the form $c_{ij} = c + \gamma_k$, where c is known and $-1 < \gamma_k < 1$ ($k \in \{1, 2, 3\}$) is an unknown perturbation, are displayed as thin solid lines with white arrows (and k crossing bars before the tip).

bounded perturbation, namely,

$$A = \overbrace{\begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}}^{A_n} + \overbrace{\begin{pmatrix} -\Delta_1 & \Delta_1 & 0 \\ \Delta_2 & 0 & 0 \\ 0 & 0 & -\Delta_3 \end{pmatrix}}^{\Delta A}. \tag{3.35}$$

Hence, we can construct $P = P_n + \Delta P$, where $P_n = H_n + A_n$ and $\Delta P = \Delta H + \Delta A$, and P lies within a convex set P^I, as stated in Theorem 3.8.

For the simulations we have considered $\alpha = 0.95$, hence, in order to apply Theorem 3.8, we have to verify that the LMI of (3.30) holds true for all vertices P^* of P^I. This is easily done using the LMI toolbox of Matlab, that yields

$$Z_1 = Z_2 = \begin{pmatrix} 4.008 & 0.688 & 0.400 \\ 0.688 & 1.931 & -0.114 \\ 0.440 & -0.114 & 3.509 \end{pmatrix},$$

both of them positive definite. Therefore, Theorem 3.8 predicts that H in (3.34) can be used as a (local) synchronizer between the networks. When the nominal parameter values are accurate, i.e. $\Delta A = 0$ and $\Delta H = 0$ (no perturbation), the networks attain identical synchronization very quickly, as shown by Fig. 3.3. To numerically assess the robustness of proposed scheme and corroborate the analytical results in Section 3.5.2, we have generated the perturbations shown in Fig. 3.4. They change frequently over time, but remain bounded between -1 and $+1$.

The synchronization errors are displayed in Fig. 3.5. This plot shows that changes in the parameters of the matrices H and A result in small perturbations of the synchronization errors, which are quickly damped. It also reveals that the synchronization errors do not exhibit any response to variations in the matrix C, as long as it remains diffusive.

Fig. 3.6 is a zoom into Fig. 3.5 that shows the transient behavior of the synchronization errors and the magnitude of the error perturbations due to the parameter changes.

3.6 Summary and conclusions

We have addressed the problem of outer synchronization between two networks with fractional-order dynamics. Starting from a basic result that introduces the notion of α-stability for fractional-order systems, we obtain necessary and sufficient conditions for outer synchronization in terms of the

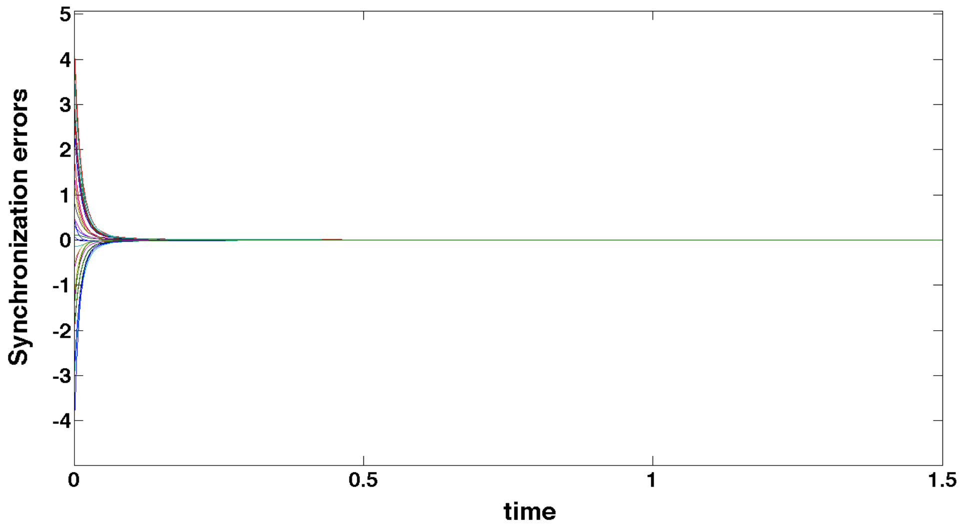

Figure 3.3: Convergence toward zero of the outer synchronization errors with the same values of Fig. 3.4, when there is no perturbation.

relative positions of the eigenvalues of a certain matrix that governs the error dynamics. To be specific, we provide sufficient and necessary conditions for the fractional-order differential equation of the synchronization error, $\overline{e}(t)$, to have a fixed point at $\overline{e}(t) = 0$.

The assumptions that we impose on the structure of the coupling matrix of the networks are relatively mild. In particular, for a first set of results, we only assume that the coupling is symmetric and diffusive. Since integer-order differential equations are just a particular case of fractional-order equations, our analysis is also valid for the outer synchronization of networks with ordinary (integer order) dynamics, as explicitly shown by Theorem 3.2 and Corollary 3.1.

We have also introduced a new set of conditions for outer synchronization given in terms of LMI's. Such conditions are often easier to check than eigenvalue positions. To be specific, we have introduced different sets of LMI's for different ranges of the fractional order α, under the assumption of diffusive coupling. The approach based on LMI's is flexible enough to be extended to cases in which the parameters of the networks are only known up to bounded perturbations. Assuming the latter bound is known and the coupling matrix of the networks is diffusive, we have also found conditions for synchronization (this is termed "robust synchronization" because it holds

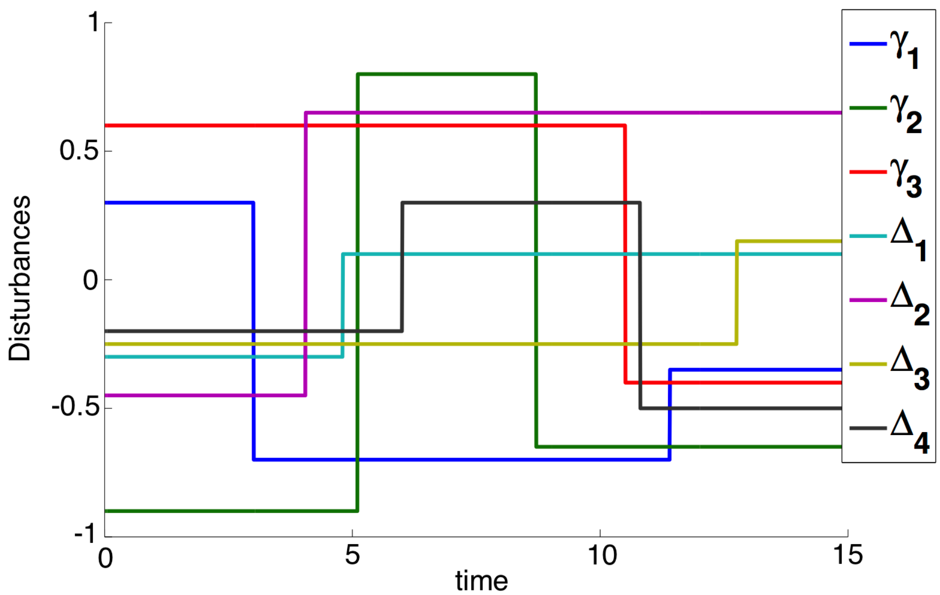

Figure 3.4: Time-varying perturbations of the parameters in Eqs. (3.32), (3.33) and (3.34).

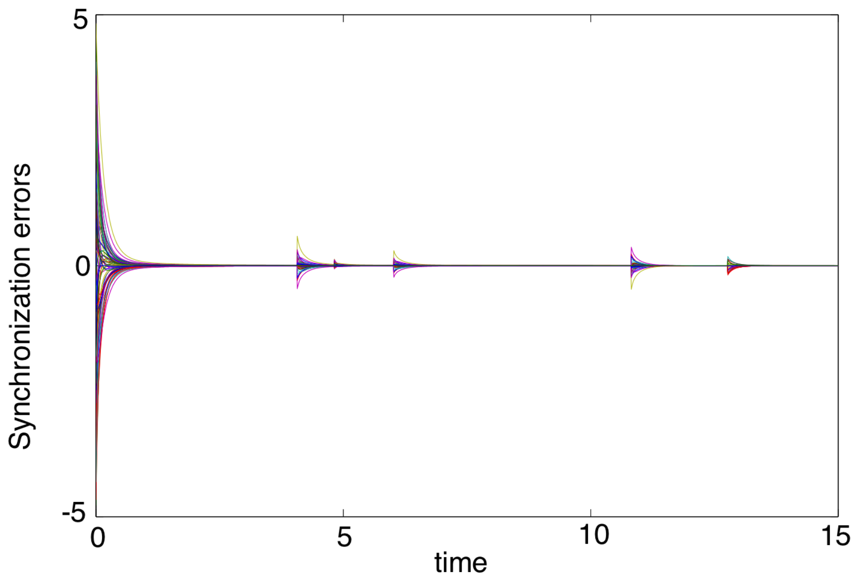

Figure 3.5: Convergence toward zero of the outer synchronization errors, $\overline{e}_i(t)$, for a system with $N = 23$ nodes per network, Lorenz fractional-order dynamics, $\alpha = 0.95$ and bounded perturbations.

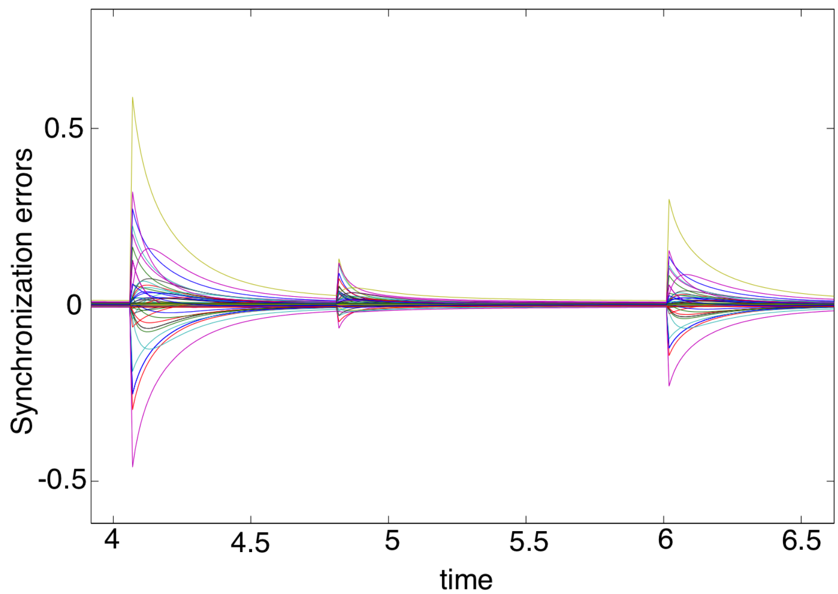

Figure 3.6: Transient behavior of the outer synchronization errors. This plot is a zoom into Fig. 3.4.

no matter the exact values of the perturbations, as far as they are bounded).

Finally, we have also investigated the generalized outer synchronization of the networks. In particular, we have shown that it can be attained with a very simple scheme, provided that the coupling matrix is diffusive and irreducible. To complement it, we have also presented a numerical simulation that shows how generalized synchronization is actually attained as suggested by the convergence of the approximate errors.

The numerical study also includes simulation results for the robust synchronization of two networks with perturbed parameters. Specifically, we have shown how two coupled networks with $N = 23$ nodes each, scale-free structure and fractional-order Lorenz dynamics can be synchronized even when the parameters of the differential equations and the coupling matrix suffer a (bounded) perturbation.

Chapter 4

Synchronization of ordinary networks

4.1 Introduction

Despite the increasing interest of researchers in the study of complex dynamical networks in recent years, the amount of analytical results related to outer synchronization that can be found in the literature is still limited. In [64] and [65], synchronization between two continuous-time [64] and discrete-time [65] complex networks in a master-slave configuration is investigated using similar approaches. In both cases, the coupling matrix that determines the topology of the networks is assumed to be diffusive and the design of the synchronizers is based on the calculation of Jordan canonical forms. Only local synchronization is guaranteed (as a result of using a linearization scheme) and the master and slave systems have to be fully deterministic. No unknown perturbations of the network variables or parameters is considered. Another example of using linearization techniques to attain local outer synchronization is [69]. Similar to [64, 65], the networks are assumed to be fully deterministic (no parameter mismatch or dynamical disturbances are considered) and the coupling matrix diffusive and, additionally, balanced.

The work in [124] is concerned with attaining outer synchronization in finite time between two networks whose state variables are perturbed by an additive Brownian motion process. The networks are assumed to have the same node dynamics but possibly different topological structure. The same as in the previous references, the coupling matrices are assumed to be diffusive. An important feature of the scheme in [124] is the necessity to gather signals from all the nodes in the master network in order to compute the synchronizer for each individual node in the slave network. For large networks, this feature may arise obvious difficulties with the practical complexity of the scheme. The model parameters are also considered to be deterministic and known, the same as in the former references.

Chapter 3 in this thesis provides an example of a scheme where synchronization is robust to perturbations in the model parameters, which can be different and unknown across different network nodes and across different networks. Moreover, some of the schemes proposed in Chapter 3 are proved to be valid for networks with non-diffusive coupling matrices. However, similar to [64, 65, 69], these results rely on the use of linearization techniques and, hence, they can only guarantee local outer synchronization.

In this chapter we investigate robust schemes for global outer synchronization of two diffusively-coupled complex dynamical networks in which the model parameters are not known, i.e., they are subject to an unknown perturbation with respect to their nominal values. Our approach relies on

(a) a basic lemma on the eigendecomposition of matrices resulting from Kronecker products and

(b) a suitable choice of a Lyapunov function related to the synchronization error dynamics.

Starting from these two ingredients, a theorem that provides a sufficient condition for the global outer synchronization of two networks with known parameters and a diffusive coupling matrix[1] is proved. The sufficient condition in the latter theorem is formally given as an LMI that has to be satisfied by the system of coupled networks[2]. The argument of the proof includes the design of the gain of the synchronizer, which is a constant square matrix with dimension given by the number of dynamic variables in a single network node. Therefore, the complexity of the scheme is independent of the size of the overall network, which can be much larger.

The basic result is subsequently elaborated, first in order to simplify the design of the synchronizer while holding the assumption of the coupling matrix being diffusive. Then, the latter assumption is relaxed and

[1]It should be remarked that the term "coupling matrix" here refers to the matrix that specifies the inner connections of each network. This is different from the network-to-network coupling scheme, which is *not* given by the coupling matrix but ideally has to be designed to ensure synchronization. Specifically, the fact that the network-to-network coupling is diffusive is independent of the (intra-network) coupling matrices being diffusive.

[2]Most of the subsequent results that stem from this theorem are also expressed in terms of LMI's that need to be satisfied.

a sufficient condition for global outer synchronization is given. The corresponding LMI involves the maximum eigenvalue of the coupling matrix but avoids any other assumptions on it (in particular, the coupling matrix is not assumed to be diffusive anymore). Next we investigate schemes that reduce the dimension of the synchronizer signals, which can be made lesser than the dimension of the state in a single node. Finally, we obtain synchronizers that are robust to model errors in the parameters of the networks. As before, sufficient conditions for global synchronization are given in the form of an LMI with only mild assumptions on the coupling matrix. An illustrative numerical example for the outer synchronization of two networks of classical Lorenz nodes with perturbed parameters is presented.

The rest of this chapter is organized as follows. In Section 4.2 we present a formal description of the network model and a statement of the synchronization problem to be addressed. A set of auxiliary results, are presented in Section 4.3. In Section 4.4 we introduce the main analytical results. A numerical example is presented in Section 4.5 and, finally, Section 4.6 is devoted to the conclusions.

4.2 Network model and problem statement

4.2.1 Network model

The network models used in this chapter are essentially the same as those in Chapter 3; only the fractional orders on the left hand side of equations (3.1) and (3.2) are replaced by integer ones. The master and slave networks in this chapter are, therefore, formally described as

$$\dot{x}_i = Ax_i + f\left(x_i\right) + \sum_{j=1}^{N} c_{ij} L x_j, \quad i = 1, \ldots, N, \tag{4.1}$$

and

$$\dot{y}_i = Ay_i + f\left(y_i\right) + \sum_{j=1}^{N} c_{ij} L y_j + u_i, \tag{4.2}$$

respectively, where $u_i(t) \in \mathbb{R}^n$ is the synchronizer signal, defined as $u_i = K(y_i - x_i)$, and K is a constant matrix which will be later designed in such a way that outer synchronization, as described by Definition 2.10 of Section 2.6.2, can be guaranteed.

Matrix C is a key element for characterizing the dynamics of both different equations (4.1) and (4.2). In the literature, various assumptions are commonly made in order to simplify the analysis of the class of systems described by Eq. (4.1). The most common assumptions are symmetry, irreducibility, balance and, in particular, diffusivity. These properties are described in Section 2.6.3

In this Chapter, we relax all of these assumptions. Indeed, we show that appropriate synchronization schemes can be found without assuming diffusivity, symmetry, balance and irreducibility.

In the sequel, we assume that the matrix L in Eqs. (4.1) and (4.2) is an $n \times n$ identity matrix, $L = I_n$. This is done for the sake of clarity in the presentation of the introduced results, but they can be extended for other values of L.

4.2.2 Problem statement

Let us introduce the $n \times 1$ error signal $e_i(t) = y_i(t) - x_i(t) \in \mathbb{R}^n$. The error dynamics are described by the differential equation

$$\begin{aligned} \dot{e}_i &= \dot{y}_i - \dot{x}_i \\ &= Ae_i + f(y_i) - f(x_i) + \sum_{j=1}^{N} c_{ij} e_j + u_i, \end{aligned} \tag{4.3}$$

where the synchronizer signal can be rewritten as $u_i = K e_i$.

Stacking together the error signals for the N pairs of nodes in the overall system, we can define the $nN \times 1$ global error vector

$$e(t) = \left[e_1^\top(t), e_2^\top(t), \ldots, e_N^\top(t)\right] \in \mathbb{R}^{nN}, \tag{4.4}$$

where the superscript $\top$ denotes transposition, and the resulting error dynamics can be compactly written as

$$\dot{e} = (I_N \otimes (A + K) + C \otimes I_n)\, e + \bar{f}, \tag{4.5}$$

where

$$\bar{f} = \bar{f}(t) = \left[(f(y_1(t)) - f(x_1(t)))^\top, \ldots, (f(y_N(t)) - f(x_N(t)))^\top\right]^\top \tag{4.6}$$

is the $nN \times 1$ vector collecting the synchronization error in the nonlinear components of the nodes. Note that we also skip the time dependence of $\bar{f}(t)$ in Eq. (4.5).

Our goal in this chapter is to find sufficient conditions for the two networks to synchronize globally, i.e., to ensure that $\lim_{t\to\infty} \|e(t)\| = 0$ irrespective of the initial conditions. Note that, since the matrices A and C and the nonlinear function f are given, synchronization has to be achieved by a proper design of the gain matrix K.

4.3 Ancillary results

Studying the global synchronization of the networks (4.1) and (4.2) amounts to analyzing the global asymptotic stability of the nN-dimensional error signal $e(t)$, whose dynamics are determined by Eq. (4.5). Since the number of nodes in the network, N, can be very large in practical applications, matrix-vector calculations involving $e(t)$ (and the coupling matrix C as well) may turn out prohibitive and there is a need to find analytical methods which are both rigorous and efficient. In this section we review a number of auxiliary results that will be later used to alleviate this difficulty.

Lemma 3.1 introduced in Section of Chapter 3 has a key role of synchronization analysis in this Chapter. In the sequel, some other results utilized in proofs of this chapter are brought.

The following lemma states that the eigenvalues of a matrix A are shifted by a constant k when we perform the operation $A + kI$.

Lemma 4.1 *Let $\eta_i, i = 1, \ldots, n$ be the eigenvalues of the $n \times n$ matrix A. The eigenvalues of matrix $A+kI_n$, where $k \in \mathbb{R}$ is an arbitrary real constant, are*

$$\xi_i = \eta_i + k, \quad i = 1, \ldots, n. \tag{4.7}$$

The next lemma states the well-known Lyapunov stability condition and makes a connection among the eigenvalues of a matrix, asymptotic system stability and the satisfaction of a corresponding LMI.

Proof. According to the definition of eigenvalues of Matrix A, the proof is straightforward.

Lemma 4.2 *Let A be an $n \times n$ matrix and let $x(t) \in \mathbb{R}^n$ be a dynamic vector. The following statements are equivalent:*

- *The linear system $\dot{x} = Ax$ is asymptotically stable around $x_o = 0$.*
- *All the eigenvalues of matrix A have negative real part.*

- *There exists a positive definite matrix P such that the LMI*

$$PA + A^\top P < 0$$

is satisfied.

Proof. See [77].

As mentioned in Section 4.1, most of the earlier work on the outer synchronization of complex dynamical networks builds upon the assumption that the coupling matrix C that describes the intra-network topology is diffusive. The reason is that this property entails a number of other useful results involving the eigenvalues of C, which are restated by the next lemma.

Lemma 4.3 *All the eigenvalues of a diffusive matrix C have nonpositive real parts. Moreover,* 0 *is an eigenvalue of C in general and, if C is irreducible, then* 0 *is an eigenvalue with multiplicity one.*

Proof. See [121].

Finally, the inequality below will be ancillary for the stability analysis of the error dynamics (4.5) using Lyapunov functions.

Lemma 4.4 *Choose arbitrary matrices A and B with compatible dimensions. The inequality*

$$A^\top B + B^\top A \leq A^\top Q A + B^\top Q^{-1} B$$

is satisfied for any positive definite square matrix Q with suitable dimensions.

Proof. See [13].

4.4 Global synchronization

In this section we introduce sets of sufficient conditions, expressed as LMI's that involve the gain matrix K, for global synchronization of the networks (4.1) and (4.2). We start, in Section 4.4.1, with basic results for systems where the model parameters are exactly known. We analyze the cases in which the coupling matrix is diffusive (the same as in the existing literature) and then relax this assumption. We also seek schemes where the dimension of the synchronizer signals can be reduced (namely, where it can be made smaller than the state space dimension n) in Section 4.4.2. In Section 4.4.3 we extend our analysis to systems of coupled networks where the model parameters are subject to an unknown (albeit bounded) perturbation.

4.4.1 Main results

Recall that the local synchronizer signals at the network nodes have the form $u_i(t) = Ke_i(t)$, where $e_i(t) = y_i(t) - x_i(t)$ is the synchronization error at the i-th node and K is a constant (network wide) gain matrix of dimension $n \times n$, and $\eta > 0$ is the Lipschitz constant of the nonlinear function f. The following theorem provides a sufficient condition for the gain matrix K to guarantee the global synchronization of networks (4.1) and (4.2) when the coupling matrix C is diffusive. This is a fundamental result that allows several extensions of the analysis as the assumptions on the model are changed.

Theorem 4.1 *Assume that the coupling matrix C is diffusive. If there exist a symmetric and positive definite matrix $X \in \mathbb{R}^{n\times n}$ and a positive definite matrix $W \in \mathbb{R}^{n\times n}$ such that the LMI*

$$\left(A + \frac{1+\eta^2}{2} I_n\right) X + X \left(A + \frac{1+\eta^2}{2} I_n\right)^\top + W + W^\top < 0 \tag{4.8}$$

is satisfied, then the gain matrix $K = WX^{-1}$ guarantees that the networks (4.1) and (4.2) synchronize globally.

Proof. Consider the radially-unbounded Lyapunov function

$$V = e^\top \tilde{P} e \tag{4.9}$$

where $\tilde{P}$ is some positive definite matrix of dimension $nN \times nN$. If $\dot{V} < 0$, $\forall e \neq 0$, then the system of differential equations (4.5) is asymptotically stable around $e(t) = 0$ (irrespective of its initial condition). Therefore, it is enough to show that (4.8) implies $\dot{V} < 0$ when the gain matrix is selected as $K = X^{-1}W$, and we proceed to prove the latter result.

The derivative $\dot{V}$ can be easily obtained as

$$\dot{V} \quad = e^\top \tilde{P}\dot{e} + \dot{e}^\top \tilde{P} e. \tag{4.10}$$

If we denote, for conciseness,

$$\begin{aligned} Y &= I_N \otimes (A + K), \quad \text{and} & (4.11)\\ Z &= C \otimes I_n, & (4.12) \end{aligned}$$

then substituting Eq. (4.5) into Eq. (4.10) yields

$$\dot{V} = e^\top \left(\tilde{P}\,(Y+Z) + (Y+Z)^\top \tilde{P}\right) e + e^\top \tilde{P}\bar{f} + \bar{f}^\top \tilde{P} e.$$

Moreover, according to Lemma 4.4, we may choose an arbitrary positive definite matrix Q of dimension $nN \times nN$ to obtain the inequality

$$\dot{V} \leq \left(\tilde{P}\left(Y+Z\right)+\left(Y+Z\right)^{\top}\tilde{P}\right)e+e^{\top}\tilde{P}Q\tilde{P}e+\bar{f}^{\top}Q^{-1}\bar{f}$$

and, using the fact that $\|\bar{f}\| < \eta\|e\|$ (that follows from f being Lipschitz with constant $\eta > 0$), we arrive at

$$\dot{V} \leq e^{\top}\left(\tilde{P}\left(Y+Z\right)+\left(Y+Z\right)^{\top}\tilde{P}+\tilde{P}Q\tilde{P}+\eta^{2}Q^{-1}\right)e. \tag{4.13}$$

Since Q is positive definite but otherwise arbitrary, we can select $Q = \tilde{P}^{-1}$ which, when substituted into (4.13), results in

$$\dot{V} \leq e^{\top}\left(\tilde{P}\left(Y+Z+\frac{1+\eta^{2}}{2}I_{nN}\right)+\left(Y+Z+\frac{1+\eta^{2}}{2}I_{nN}\right)^{\top}\tilde{P}\right)e. \tag{4.14}$$

Let us write

- $\{\mu_i; i = 1, \ldots, n\}$ for the eigenvalues of matrix $A + K$,
- $\{\lambda_j; j = 1, \ldots, N\}$ for the eigenvalues of the coupling matrix C, and
- $\{\xi_\ell; \ell = 1, ..., nN\}$ for the eigenvalues of $Y + Z + \frac{1+\eta^2}{2} I_{nN}$.

On one hand, Lemma 3.1 enables us to obtain the eigenvalues of $Y + Z$ in terms of the eigenvalues of $A + K$ and C. On the other hand, Lemma 4.1 yields the eigenvalues of $Y + Z + \frac{1+\eta^2}{2} I_{nN}$ given those of $Y + Z$. Combining both results leads to

$$\xi_\ell = \mu_i + \lambda_j + \frac{1+\eta^2}{2}, \quad \ell \in \{1, ..., nN\}, \tag{4.15}$$

where the subscript ℓ is a function of i and j, namely,

$$\ell = Ni + j, \quad \text{with} \quad i = 0, \ldots, n-1, \quad \text{and} \quad j = 1, \ldots, N.$$

The following argument brings the proof to a conclusion. Assume that the inequality (4.8) holds true. If we pre- and post-multiply by X^{-1} then we obtain the inequality

$$X^{-1}\left(A+\frac{1+\eta^{2}}{2}I_{n}\right)+\left(A+\frac{1+\eta^{2}}{2}I_{n}\right)^{\top}X^{-1}+X^{-1}WX^{-1}+X^{-1}W^{\top}X^{-1} < 0. \tag{4.16}$$

If we let $K = WX^{-1}$ and denote $P = X^{-1}$ (hence, P is symmetric and positive definite), Eq. (4.16) can be rewritten as

$$P\left(A + \frac{1+\eta^2}{2}I_n + K\right) + \left(A + \frac{1+\eta^2}{2}I_n + K\right)^\top P < 0. \tag{4.17}$$

Using Lemma 4.1, the eigenvalues of the matrix $A + \frac{1+\eta^2}{2}I_n + K$ are shown to have the form $\{\mu_i + \frac{1+\eta^2}{2}; i = 1, ..., n\}$. Moreover, from Lemma 4.2, the inequality (4.17) implies that the eigenvalues of $A + \frac{1+\eta^2}{2}I_n + K$ must have negative real parts, hence

$$\mathfrak{Real}\left\{\mu_i + \frac{1+\eta^2}{2}\right\} < 0, \quad i = 1, ..., n. \tag{4.18}$$

Since the coupling matrix C is diffusive, it follows from Lemma 4.3 that

$$\mathfrak{Real}\{\lambda_j\} \leq 0, \quad j = 1, ..., N, \tag{4.19}$$

and Eqs. (4.15), (4.18) and (4.19) together imply that

$$\mathfrak{Real}\{\xi_\ell\} < 0, \quad \ell = 1, ..., nN. \tag{4.20}$$

However, Lemma 4.2 and Eq. (4.20) show that

$$\tilde{P}\left(Y + Z + \frac{1+\eta^2}{2}I_{nN}\right) + \left(Y + Z + \frac{1+\eta^2}{2}I_{nN}\right)^\top \tilde{P} < 0, \tag{4.21}$$

and combining (4.21) and (4.14) yields $\dot{V} < 0$, $\forall e \neq 0$, which concludes the proof.

□

Let $\{\alpha_i; i = 1, ..., n\}$ be the eigenvalues of matrix A and let $a^+ = \max_{1\leq i\leq n} \mathfrak{Real}\{\alpha_i\}$ be the maximum over the real parts of these eigenvalues. The following theorem shows that, in practice, it is possible to compute very simple gain matrices that lead to global outer synchronization of the networks.

Theorem 4.2 *Assume that the coupling matrix C is diffusive and choose a real constant k satisfying the inequality*

$$k > \frac{1+\eta^2}{2} + a^+. \tag{4.22}$$

If the gain matrix is selected as $K = -kI_n$ then the networks (4.1) and (4.2) synchronize globally.

Proof. From Lemma 4.1, the eigenvalues of $A + K = A - kI_n$ have the form $\mu_i = \alpha_i - k$, $i = 1, ..., n$. Therefore, from the inequality (4.22) we easily obtain that

$$\mathfrak{Real}\left\{\mu_i + \frac{1+\eta^2}{2}\right\} = \mathfrak{Real}\{\alpha_i\} + \frac{1+\eta^2}{2} - k \le a^+ + \frac{1+\eta^2}{2} - k < 0. \quad (4.23)$$

Combining (4.23) and (4.19) we arrive at (4.20) which, in turn, leads to the inequality (4.21) and, as a consequence, $\dot{V} < 0 \; \forall e \neq 0$.

□

Theorems 4.1 and 4.2 still place the (common) assumption of diffusivity on the coupling matrix C. However, a simple combination of the latter results yields a sufficient condition for synchronization that is free of any assumption on matrix C. Specifically, let $c^+ = \max_{1\le j\le N} \mathfrak{Real}\{\lambda_j\}$ be the maximum over the real parts of the eigenvalues of the coupling matrix. The following result establishes a sufficient condition for synchronization with possibly non-diffusive matrix C.

Theorem 4.3 *If there exist a symmetric positive definite matrix X and positive definite matrix W such that the LMI*

$$\left(A + \left(\frac{1+\eta^2}{2} + c^+\right) I_n\right) X + X \left(A + \left(\frac{1+\eta^2}{2} + c^+\right) I_n\right)^\top + W + W^\top < 0 \quad (4.24)$$

is satisfied, then the gain matrix $K = WX^{-1}$ yields global synchronization of the networks (4.1) and (4.2).

Proof. Let ζ_j, $1 \le j \le n$, be the eigenvalues of the matrix $A + K + \left(c^+ + \frac{1+\eta^2}{2}\right) I_n$. From Eq. (4.15), Lemma 4.1 and the definition of c^+, it follows that

$$\max_{1\le\ell\le nN} \mathfrak{Real}\{\xi_\ell\} \le \max_{1\le j\le n} \mathfrak{Real}\{\zeta_j\}.$$

Therefore, if $\max_{1\le j\le n} \mathfrak{Real}\{\zeta_j\} < 0$ then $\max_{1\le\ell\le nN} \mathfrak{Real}\{\xi_\ell\} < 0$ which, in turn, implies $\dot{V} < 0 \; \forall e \neq 0$ and global synchronization.

Lemma 4.2 shows that $\max_{1\le j\le n} \mathfrak{Real}\{\zeta_j\} < 0$ if, and only if,

$$P\left(A + K + \left(c^+ + \frac{1+\eta^2}{2}\right) I_n\right) + \left(A + K + \left(c^+ + \frac{1+\eta^2}{2}\right) I_n\right)^\top P < 0 \quad (4.25)$$

for some positive definite matrix P. However, if the LMI (4.24) holds true and we choose $P = X^{-1}$ and $W = KX$ (hence, $K = WX^{-1}$), then (4.25) follows immediately, which concludes the proof. □

Remarkably, it is possible to apply Theorem 4.3 without explicitly calculating the eigenvalues of matrix C. Note that the latter may be a computationally heavy task in practice, since it can be expected that $N >> n$. To avoid such calculation, we resort to the following auxiliary lemma.

Lemma 4.5 (Gershgorin circle theorem) *Let A be a complex $n \times n$ matrix with entries a_{ij}, and let $\mathcal{D}(a_{ii}, \varsigma_i)$ be the closed disc centered at a_{ii} with radius $\varsigma_i = \sum_{j \neq i} |a_{ij}|$. All eigenvalues of A lie in at least one of the Gershgorian discs $\mathcal{D}(a_{ii}, \varsigma_j)$, $1 \leq i \leq n$.*

Proof. See [47].

Let $\bar{c} = \max_{1 \leq i \leq N} \left| c_{ii} + \sum_{j \neq i} |c_{ij}| \right|$. The following corollary provides a simpler way of applying Theorem 4.3 in practical setups.

Corollary 4.1 *If there exist a symmetric positive definite matrix X and positive definite matrix W such that the LMI*

$$\left(A + \left(\frac{1+\eta^2}{2} + \bar{c} \right) I_n \right) X + X \left(A + \left(\frac{1+\eta^2}{2} + \bar{c} \right) I_n \right)^\top + W + W^\top < 0 \tag{4.26}$$

is satisfied, then the gain matrix $K = WX^{-1}$ yields global synchronization of the networks (4.1) and (4.2).

Proof. From Lemma 4.5, it is seen that $c^+ \leq \bar{c}$. The result is then straightforward from Theorem 4.3.

□

4.4.2 Lower dimensional gain matrix

In many cases, the networks may not be coupled through the full dimensional state signals $x_i(t) \in \mathbb{R}^n$, $i = 1, ..., N$. In control problems, in which the slave system modeled by network (4.2) has to be steered by the master system modeled by (4.1), this may be due to physical limitations of the actuators that should apply the control signals. In other scenarios, there may be real-world constraints in the manner the two systems modeled by the networks (4.1) and (4.2) can interact.

In order to account for such limitations, we can substitute the synchronizer signals $u_i = K(y_i - x_i)$ in network (4.2) by $\bar{u}_i = B\bar{K}(y_i - x_i)$, $i = 1, ..., N$, where $B \in \mathbb{R}^{n \times m}$ is a given tall matrix, i.e., $m < n$, and $\bar{K} \in \mathbb{R}^{m \times n}$ is a short gain matrix, whose dimension is reduced with respect to the original matrix $K \in \mathbb{R}^{n\times n}$. As a result of this substitution, the error dynamics of Eq. (4.5) becomes

$$\dot{e} = \big(I_N \otimes (A + B\bar{K}) + C \otimes I_n\big)\, e + \bar{f}. \tag{4.27}$$

It turns out that it is possible to carry out the same kind of global stability analysis that led to Theorem 4.1 when the networks are coupled through the lower-dimensional gain matrix $\bar{K}$.

Theorem 4.4 *Assume that the coupling matrix C is diffusive and $B \in \mathbb{R}^{n\times m}$, $m \le n$. If there exist $W \in \mathbb{R}^{m\times n}$ and a symmetric and positive definite matrix $X \in \mathbb{R}^{n\times n}$ such that the LMI*

$$\left(A + \frac{1+\eta^2}{2} I_n\right) X + X \left(A + \frac{1+\eta^2}{2} I_n\right)^{\top} + BW + W^{\top}B^{\top} < 0 \tag{4.28}$$

is satisfied, then the gain matrix $\bar{K} = WX^{-1}$ guarantees that the networks (4.1) and (4.2) synchronize globally.

Proof. The proof is essentially the same as for Theorem 4.1. In particular, we can work with exactly the same radially unbounded Lyapunov function V and arrive at the inequality (4.14). Then we write $\{\lambda_j; j = 1, ..., N\}$ for the eigenvalues of C, $\{\xi_\ell; \ell = 1, ..., nN\}$ for the eigenvalues of $Y + Z + \frac{1+\eta^2}{2} I_{nN}$ but introduce $\{\bar{\mu}_i; i = 1, ..., n\}$ for the eigenvalues of $A + B\bar{K}$. Then, from Lemmas 3.1 and 4.1 we obtain

$$\xi_\ell = \bar{\mu}_i + \lambda_j + \frac{1+\eta^2}{2}, \quad \ell \in \{1, ..., nN\}, \tag{4.29}$$

where, $\ell = Ni + j$, with $i = 0, \dots, n-1$, and $j = 1, \dots, N$, which is the straightforward counterpart of Eq. (4.15).

Finally, assume that (4.28) holds true. If we pre- and post-multiply by X^{-1} in (4.28) and then let $\bar{K} = WX^{-1}$ and $P = X^{-1}$ we obtain

$$P\left(A + \frac{1+\eta^2}{2} I_n + B\bar{K}\right) + \left(A + \frac{1+\eta^2}{2} I_n + B\bar{K}\right)^{\top} P < 0. \tag{4.30}$$

From (4.30), we can use Lemmas 4.1 and 4.2 in exactly the same way as in the proof of Theorem 4.1 to show that $\mathfrak{Real}\{\xi_\ell\} < 0$, $\ell = 1, ..., nN$. The latter inequality combined with Lemma 4.2 brings us to Eq. (4.21) again which, combined with Eq. (4.14), yields $\dot{V} < 0\ \forall e \neq 0$.

$\square$

Theorem 4.4 guarantees that, if the matrices $W \in \mathbb{R}^{m\times n}$ and $X > 0$ exist, then global synchronization is attained with the reduced-dimension gain matrix $\bar{K} = WX^{-1}$. However, depending on the pair of matrices $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, it may happen that no $X > 0$ and W exist that satisfy the LMI (4.28). This difficulty can be removed if we assume the pair of matrices (A, B) to be controllable [87].

Definition 4.1 *The pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, $m \le n$, is controllable if the $n \times nm$ real matrix $\left[B, AB, \cdots, A^{(n-1)}B\right]$ has rank n.*

If (A, B) is controllable, then we can select the eigenvalues of the sum $A+B\bar{K}$ by adequately choosing $\bar{K}$.

Lemma 4.6 *A pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is controllable if, and only if, for any valid[3] choice $\mathcal{U} = \{\nu_i; i = 1, ..., n\}$ there exists $\bar{K}_U \in \mathbb{R}^{m\times n}$ such that $\mathcal{U}$ is the set of eigenvalues of the sum matrix $A + B\bar{K}_U$.*

Proof. See [87, pages 829-832].

Therefore, the pair (A, B) being controllable is actually a sufficient condition for global synchronization of the networks (4.1) and (4.2).

Theorem 4.5 *Assume that the coupling matrix C is diffusive and $B \in \mathbb{R}^{m\times n}$, $m \le n$, is such that the pair (A, B) is controllable. Then, there exists $\bar{K} \in \mathbb{R}^{m\times n}$ such that the networks (4.1) and (4.2) synchronize globally.*

Proof. Recall that $\{\bar{\mu}_i; i = 1, ..., n\}$ denotes the set of eigenvalues of $A + B\bar{K}$. From Lemma 4.1, the eigenvalues of $A + B\bar{K} + \frac{1+\eta^2}{2} I_n$ have the form $\nu_i = \mu_i + \frac{1+\eta^2}{2}$, $i = 1, ..., n$. However, if (A, B) is controllable, then, from Lemma 4.6, there exists $\bar{K}$ such that $\mathfrak{Real}\{\mu_i\} < -\frac{1+\eta^2}{2}$, and hence $\mathfrak{Real}\{\nu_i\} < 0$, for $i = 1, ..., n$.

From Lemma 4.2, $\mathfrak{Real}\{\nu_i\} < 0$ for $i = 1, ..., n$ implies that there exists some positive definite and symmetric $P \in \mathbb{R}^{n\times n}$ such that

$$P\left(A + B\bar{K} + \frac{1+\eta^2}{2} I_n\right) + \left(A + B\bar{K} + \frac{1+\eta^2}{2} I_n\right)^\top P < 0. \qquad (4.31)$$

If we choose $X = P^{-1}$ and $W = \bar{K}P^{-1}$ then it is seen that (4.31) is equivalent to the LMI (4.28) in the statement of Theorem 4.4.

□

[3]The elements of $\mathcal{U}$ must either be real or appear in conjugate pairs in order to be valid eigenvalues of a real matrix.

Remark 4.1 *A gain matrix $\bar{K} \in \mathbb{R}^{m\times n}$ that guarantees synchronization can be found by computing $X > 0$ and W that satisfy the LMI (4.28). Theorem 4.5 simply guarantees that $X > 0$ and W exist.*

Remark 4.2 *It is straightforward to extend Theorem 4.4, in the same way as we have done with Theorem 4.1, to obtain results analogous to Theorems 4.2 and 4.3 and Corollary 4.1 when the networks are linked through the signals $\bar{u}_i(t) = B\bar{K}(y_i(t) - x_i(t)) \in \mathbb{R}^m$, $m < n$. In particular, if $\bar{c} = \max_{1\le i\le N} \left| c_{ii} + \sum_{i\neq j} |c_{ij}| \right|$ and there exist $X > 0$ symmetric and $W \in \mathbb{R}^{m\times n}$ such that the LMI*

$$\left(A + \left(\frac{1+\eta^2}{2} + \bar{c}\right) I_n\right) X + X \left(A + \left(\frac{1+\eta^2}{2} + \bar{c}\right) I_n\right)^\top + BW + W^\top B^\top < 0 \tag{4.32}$$

is satisfied, then the gain matrix $\bar{K} = WX^{-1}$ yields global synchronization of the networks (4.1) and (4.2).

4.4.3 Robust synchronization

Very often, the fixed parameters of the networks, including the gain matrix K, cannot be known exactly and the system dynamics may be subject to external unknown perturbations. Such difficulties may typically arise from modeling errors or, simply, from the impossibility to characterize a physical or otherwise real-world system faithfully enough. For this reason, it is highly desirable to determine whether a synchronization scheme can be *robust*, i.e., whether global synchronization can be guaranteed despite such model mismatches and/or perturbations.

In this section, we assume that the matrices A and K appearing in Eqs. (4.1) and (4.2) are only available up to an unknown, bounded but possibly time-varying, mismatch. Additionally, we further introduce unknown additive perturbations that affect the slave network. The latter disturbance can be arbitrary, but bounded as well.

To be precise, the master and response networks are modelled as

$$\dot{x}_i = (A + \Delta A)\, x_i + f(x_i) + \sum_{j=1}^{N} c_{ij} x_j \tag{4.33}$$

and

$$\dot{y}_i = (A + \Delta A)\, y_i + f(y_i) + \sum_{j=1}^{N} c_{ij} y_j + B\left(\bar{K} + \Delta\bar{K}\right)(y_i - x_i) + d_i(y_i - x_i), \tag{4.34}$$

respectively, for $i = 1, ..., N$, where $A \in \mathbb{R}^{n\times n}$ and $\bar{K} \in \mathbb{R}^{n\times m}$ are nominal (known) values, while $\Delta A(t) \in \mathbb{R}^{n\times n}$ and $\Delta\bar{K}(t) \in \mathbb{R}^{m\times n}$ are unknown perturbations of the nominal values, and $d_i(t)(y_i - x_i)$ is an additional disturbance of the i-th slave nodes, also unknown, with $d_i(t) \in \mathbb{R}$. Combining Eqs. (4.33) and (4.34), the dynamics of the overall synchronization error $e(t) \in \mathbb{R}^{nN}$ is shown to be governed by the differential equation

$$\dot{e} = (I_N \otimes ((A + \Delta A) + B\,(K + \Delta K)) + C \otimes I_n)\, e + \bar{f} + De, \tag{4.35}$$

where

$$D = \begin{pmatrix} d_1 I_n & & \\ & \ddots & \\ & & d_N I_n \end{pmatrix} \in \mathbb{R}^{nN\times nN} \tag{4.36}$$

is a diagonal unknown disturbance matrix.

In order to study whether the perturbed networks (4.33) and (4.34) synchronize globally, i.e., whether $\lim_{t\to\infty} e(t) = 0$, we constrain the perturbations $\Delta A(t)$, $\bar{K}(t)$ and $d(t)$ to be uniformly bounded from above over time. In particular, we assume that there exist finite constants $\gamma > 0$, $\kappa > 0$ and $\tau > 0$ such that

$$\begin{aligned} \|\Delta A(t)\| &< \gamma, && (4.37)\\ \|B\Delta K(t)\| &< \kappa \quad \text{and} && (4.38)\\ \|D\| &< \tau. && (4.39) \end{aligned}$$

If these upper bounds hold, it is straightforward to obtain an analog of Theorems 4.1 and 4.4 for the perturbed system of Eq. (4.35).

Theorem 4.6 *Assume that the coupling matrix C is diffusive, $B \in \mathbb{R}^{n\times m}$ is given and the inequalities (4.37), (4.38) and (4.39) hold. If there exist $X \in \mathbb{R}^{n\times n}$, positive definite and symmetric, and $W \in \mathbb{R}^{m\times n}$ such that the LMI*

$$\begin{aligned} &\left(A + \frac{4+\gamma^2+\kappa^2+\eta^2+\tau^2}{2} I_n\right) X + \\ &X\left(A + \frac{4+\gamma^2+\kappa^2+\eta^2+\tau^2}{2} I_n\right)^\top + BW + W^\top B^\top < 0 \end{aligned} \tag{4.40}$$

is satisfied, then the gain matrix $\bar{K} = WX^{-1}$ guarantees that the networks (4.33) and (4.34) synchronize globally.

Proof. Let us consider again the radially unbounded Lyapunov function of Eq. (4.9). If we substitute (4.35) into Eq. (4.10) and then apply the bounds (4.37)-(4.39), we readily obtain (by way of Lemma 4.4, exactly the same as in the proof of Theorem 4.1)

$$\begin{aligned} \dot{V} \quad < \quad & e^\top \left(\tilde{P} \left(Y + Z + \frac{4 + \eta^2 + \gamma^2 + \kappa^2 + \tau^2}{2} I_{nN} \right) \right. \\ & \left. + \left(Y + Z + \frac{4 + \eta^2 + \gamma^2 + \kappa^2 + \tau^2}{2} I_{nN} \right)^\top \tilde{P} \right) e, \end{aligned} \qquad (4.41)$$

which is the counterpart of the inequality (4.14), the only difference being the (larger) constant $\frac{4+\eta^2+\gamma^2+\kappa^2+\tau^2}{2}$ instead of $\frac{1+\eta^2}{2}$ obtained in the absence of perturbations. It is now straightforward to follow the argument in the proofs of Theorems 4.1 and 4.4 to complete the proof.

□

Remark 4.3 *Theorem 4.6 can be extended in the same manner as Theorems 4.1 and 4.4 to account for networks with non-diffusive coupling matrix C. In particular, if $\bar{c} = \max_{1 \le i \le N} \left| c_{ii} + \sum_{i \ne j} |c_{ij}| \right|$ and there exist $X > 0$ symmetric and $W \in \mathbb{R}^{m \times n}$ such that the LMI*

$$\begin{aligned} & \left(A + \left(\frac{4 + \gamma^2 + \kappa^2 + \eta^2 + \tau^2}{2} + \bar{c} \right) I_n \right) X + \\ & X \left(A + \left(\frac{4 + \gamma^2 + \kappa^2 + \eta^2 + \tau^2}{2} + \bar{c} \right) I_n \right)^\top + BW + W^\top B^\top < 0 \end{aligned} \qquad (4.42)$$

is satisfied, then the gain matrix $\bar{K} = WX^{-1}$ yields global synchronization of the networks (4.33) and (4.34).

Moreover, the existence of W and X that satisfy (4.42) (and, hence, the existence of the gain matrix $\bar{K} = WX^{-1}$) is guaranteed whenever the pair (A, B) is controllable.

4.5 Numerical simulations

In this section we present computer simulation results that illustrates the application of Theorem 4.6 in Section 4.4.3. In particular, we have considered the coupling of two scale-free networks, with $N = 10$ nodes each and the topology depicted in figure 4.1. The 10×10 coupling matrix that determines the connectivity of the network is

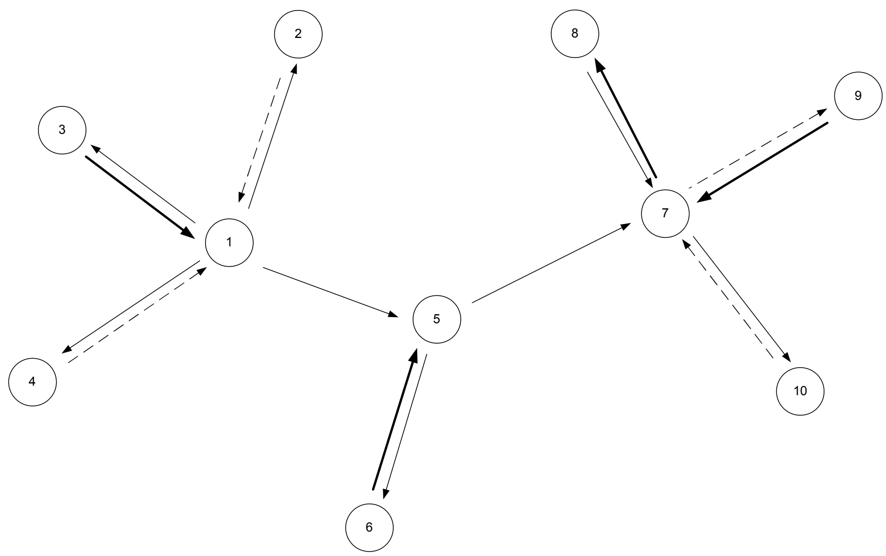

Figure 4.1: Graphical representation of the topology determined by the coupling matrix C in Eq (4.43). The links represented with thin solid lines correspond to coefficients of the form $c_{ij} = 1$; those with dashed lines correspond to coefficients of the form $c_{ij} = 2$; and the links with thick solid lines correspond to coupling coefficients of the form $c_{ij} = 3$.

$$C = \begin{pmatrix} -4 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -2 \end{pmatrix}, \qquad (4.43)$$

which can be readily checked to be diffusive. Each node in the master network corresponds to a classical 3-dimensional Lorenz system, i.e.,

$$\begin{pmatrix} \dot{x}_{i,1}(t) \\ \dot{x}_{i,2}(t) \\ \dot{x}_{i,3}(t) \end{pmatrix} = \overbrace{\begin{pmatrix} -\tilde{\sigma} & \tilde{\sigma} & 0 \\ \tilde{\rho} & -1 & 0 \\ 0 & 0 & -\tilde{\beta} \end{pmatrix}}^{(A+\Delta A)} \overbrace{\begin{pmatrix} x_{i,1}(t) \\ x_{i,2}(t) \\ x_{i,3}(t) \end{pmatrix}}^{x_i(t)} + \overbrace{\begin{pmatrix} 0 \\ -x_{i,1}(t)x_{i,3}(t) \\ x_{i,1}(t)x_{i,2}(t) \end{pmatrix}}^{f(x_i(t))}, \tag{4.44}$$

for $i = 1, ..., N$, where $\tilde{\sigma} = \sigma + \Delta_\sigma$, $\tilde{\rho} = \rho + \Delta_\rho$ and $\tilde{\beta} = \beta + \Delta_\beta$ are perturbed parameters, with nominal values $(\sigma, \rho, \beta) = (10, 28, 2/3)$ and bounded unknown perturbations $-1 \leq \Delta_\sigma, \Delta_\rho, \Delta_\beta \leq 1$. Therefore the nominal system matrix A and perturbation matrix $\Delta_A(t)$ can be written as

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix} \quad \text{and} \quad \Delta A(t) = \begin{pmatrix} -\Delta_\sigma & \Delta_\sigma & 0 \\ \Delta_\rho & 0 & 0 \\ 0 & 0 & \Delta_\beta \end{pmatrix}$$

respectively. It is relatively straightforward to compute an upper bound for $||\Delta A(t)||$. Indeed, the maximum eigenvalues of $\Delta A(t)^\top$ can be calculated directly to yield

$$\Delta_\sigma^2 + \frac{1}{2}\Delta_\rho^2 + \frac{1}{2}\sqrt{\left(4\Delta_\sigma^4 + \Delta_\rho^4\right)}.$$

Hence, from the assumption $-1 \leq \Delta_\sigma, \Delta_\rho, \Delta_\beta \leq 1$ and Lemma 4.1, we readily obtain that $||\Delta A(t)|| \leq \gamma^2 = \frac{\left(\sqrt{5}+3\right)^2}{4}$. In general, looser bounds can be obtained (in a simpler way, without explicity computing eigenvalues) via the Gershgorian circle Theorem (Lemma 4.5).

The additive disturbance in Eq. (4.34) is modeled by assuming that $d_i(t)$'s, $i = 1, \cdots, N$, are unknown random numbers in the interval $[-1, 1]$.

The Lipschitz constant for this example can be computed by taking the numerical L^2 norm of the Jacobian $J = \frac{\partial f}{\partial x}$ over the interval $[0, 1000]$, which yielded $\eta = 53$ in our simulation.

In order to interconnect the nodes in the master and slave networks, we have selected the matrix $B = [0, 1, 1]^\top$. It is easy to check that this choice makes the pair (A, B) controllable and, hence, we guarantee that the networks can be synchronized (See Theorem 4.5 and Remark 4.1). Also, since B has dimensions 3×1, we establish only one signal channel between each pair of nodes (and there is no actuation on $y_{i,1}$, $i = 1, \cdots, N$, since the first entry of B is null. This should be compared with existing schemes in

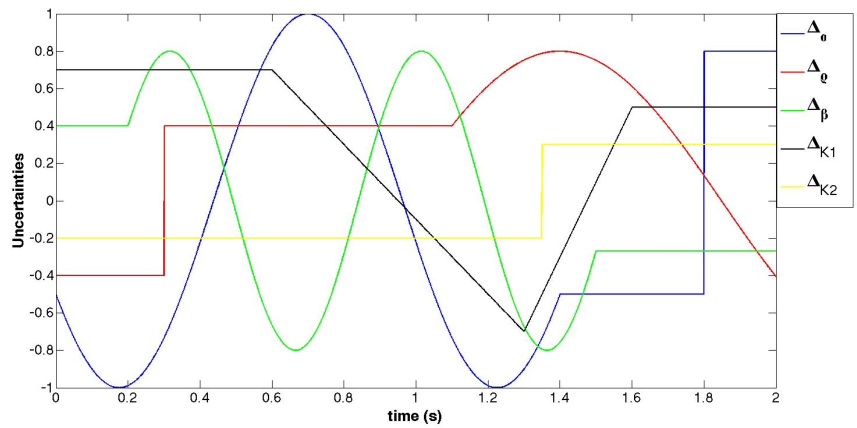

Figure 4.2: Time-varying perturbations of the parameters.

the literature. In [69], for instance, the same example is addressed but three signal channels are used and parameters have to be exactly known (there are no perturbations).

We have used the LMI toolbox of Matlab to solve the LMI of Theorem 4.6. For $B = [0, 1, 1]^\top$, this yields a nominal gain matrix $\bar{K} = [-2.6, -0.027, 0.016] \times 10^6$. We note that the magnitude of the gain coefficients is large. In particular, if this lead to any implementation difficulties, it is possible to trade off between the number of signal channels and gain amplitudes. For example, if we choose

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^\top,$$

then the nominal gain matrix can be found as

$$\bar{K} = \begin{bmatrix} -3.5 & 3.6 & -3.7 \\ 8.5 & -8.8 & 8.8 \end{bmatrix} \times 10^3$$

and, finally, choosing matrix B as

$$B_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

yields the nominal gain matrix

$$\bar{K} = \begin{bmatrix} 63 & -71 & -71 \\ -70 & -63 & 71 \\ -71 & 72 & -65 \end{bmatrix}.$$

These three pairs of matrix B and associated matrix $\bar{K}$ reveal that the more channels in control signal transmission route (i.e., less rows in matrix B), results to higher gains in elements of gain matrix. The results shown in this section correspond all of them to the choice $B_1 = [0 \quad 1 \quad 1]^\top$.

The unknown perturbations of the gain matrix is assumed to have the form

$$\Delta K(t) = \begin{pmatrix} 0 & \Delta_{K1}(t) & \Delta_{K2}(t) \end{pmatrix},$$

where $-1 \leq \Delta_{K1}(t), \Delta_{K2}(t) \leq 1$. As a result the eigenvalues of $B\Delta K(t)$ are $\{0, 0, \Delta_{K1}(t) + \Delta_{K2}(t)\}$, hence we can safely choose $\kappa^2 = 4$ as an upper bound in Eq (4.38).

Figure 4.2 displays the values of the perturbations $\Delta_\sigma(t)$, $\Delta_\rho(t)$, $\Delta_\beta(t)$, $\Delta_{K1}(t)$ and $\Delta_{K2}(t)$ over time for our simulation (recall that $d_i(t)$, $i = 1, \cdots, N$), are random, with $|d_i(t)| < 1$ as well).

Finally, Figure 4.3 plots the norm of the synchronization error, $\|e(t)\|$, versus time. We observe how $e(t)$ converges toward zero very quickly.

4.6 Summary and conclusions

We have addressed the problem of outer synchronization between two networks of nonlinear (possibly chaotic) dynamical oscillators. Our approach is based on a simple definition of the synchronization error and a proper choice of a radially-unbounded Lyapunov function. Starting from these two ingredients, we have provided sets of sufficient conditions for the global synchronization of the networks, irrespective of their initial condition. Although the first such result, Theorem 4.1, relies on relatively restrictive assumptions (diffusive coupling matrix, perfectly known network parameters), we have subsequently relaxed them to obtain sufficient conditions that ensure global synchronization when the coupling matrix is

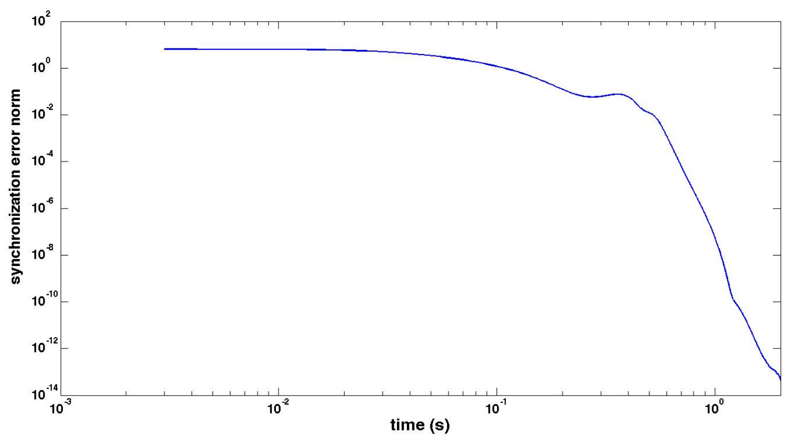

Figure 4.3: Zero convergence of error vector norm described in Eq. (4.35).

non-diffusive, when the number of connections between the two networks is reduced and when the network parameters are only known up to a bounded perturbation. In all cases, the conditions for synchronization are expressed in terms of the feasibility of an LMI whose dimension is independent of the number of nodes in the networks. Such LMI's are simple to solve using standard software and the solutions can be used explicitly to design inter-network connections that guarantee synchronization. To summarize, the key contributions of the proposed approach compared to previous work are: (a) to avoid linearizations and other approximations, hence ensuring that synchronization is attained independently of the networks initial conditions, (b) to avoid computations whose complexity depends on the network size (i.e., the number of nodes), and (c) to derive synchronization schemes that work with reduced-dimensional (even one-dimensional) signal channels between pairs of nodes. We have also provided computer simulation results that illustrate the application of the main theoretical findings in this chapter.

Chapter 5

Summary and future research

5.1 Summary

We have investigated the problem of robust synchronization between two complex dynamical networks using rigorous mathematical arguments and tools borrowed from diverse fields, such as control theory or matrix algebra.

There are two broad classes of networks, for which we have been able to obtain analytical results. The first one corresponds to networks described by differential equations of fractional-order (in Chapter 3), while the second one includes networks that can be modeled by differential equations of integer (ordinary) order (in Chapter 4). The analysis differs in the mathematical arguments, their depth and scope. A detailed summary is provided below for each scenario.

5.1.1 Networks of fractional-order oscillators

The problem of outer synchronization between two networks with fractional-order dynamics is addressed in Chapter 3. All the theorems provided in this chapter, like several other works related to the stability of fractional-order dynamical systems, are based on Matignon's stability theorem [80]. This criterion checks the stability of a linear fractional-order system based on the position of the eigenvalues of a system matrix. However, in the outer synchronization problem, the global synchronization error (and, hence, the system matrix of the error dynamics) is very large, and devising a scheme that pushes the eigenvalues into a region thet ensures the stability of the system can be a very tedious and practically impossible task. To tackle this problem, we have proposed a method to synchronize the networks without explicitly calculating the eigenvalues of the system matrix of the global error. In particular, we have found a formula to make a connection between the eigenvalues of the large system matrix with dimension $nN \times nN$ and the system matrix of local errors between each pair of nodes, which is n dimensional. Another important advantage in this method is the simple structure of the synchronization scheme, which makes it convenient for practical implementations. Several necessary and sufficient conditions have been proposed to guarantee synchronization under different assumptions.

The assumptions that we impose on the structure of the coupling matrix of the networks are relatively mild. In particular, for a first set of results, we only assume that this is symmetric and diffusive. We have also introduced a new set of conditions for outer synchronization given in terms of LMI's. Such conditions are often easier to check than eigenvalue positions. To be specific, we have introduced different sets of LMI's for different ranges of the fractional order α, under the assumption of diffusive coupling. The approach based on LMI's is flexible enough to be extended to robust synchronization, in which the parameters of the networks are only known up to bounded perturbations. Finally, we have also investigated the generalized outer synchronization of the networks. In particular, we have shown that it can be attained with a very simple scheme, provided that the coupling matrix is diffusive and irreducible.

Since integer-order differential equations are just a particular case of fractional-order equations, our analysis is also valid for the outer synchronization of networks with ordinary (integer order) dynamics. However, the existence of several strong mathematical theorems to analyze the integer-order dynamical systems, makes it possible to have better results with an even simpler synchronizer design, as shown in Chapter 4.

5.1.2 Networks of integer-order oscillators

The problem statement in Chapter 4 is basically similar to Chapter 3, but the governing dynamics of the nodes is ordinary (described by integer-order differential equations). This apparently little change in the structure of the networks makes the synchronization strategies completely different. Rather than using a linearization of the error dynamics [69], we have used the Lyapunov second stability method to guarantee the global convergence toward zero of the error norm. In particular, we have provided sets of sufficient conditions that ensure the global synchronization of the networks, irrespective of their initial condition.

The first theorem in Chapter 4 is the most restrictive one, since it requires the coupling matrix to be diffusive and the network parameters to be perfectly known. As well as in Chapter 3, the results provided in Chapter 4 are then extended to rely only on mild assumptions on the topology of the networks. We have made this possible with a novel method, consisted by suitable conversions among equal statements in the Lyapunov stability theorem. The next improvement is assuming parameters with unknown but bounded disturbances on their nominal values. Similar to Chapter 3, we have used the convexity property of LMI's to achieve the robust synchronization. Reducing the number of connections between the nodes is the last improvement that we made in the primary theorems. We showed that with the possibility of reducing the number of channels between nodes from n (number of the states in each node) to m ($1 \leq m < n$) can be checked based on the system matrix in the linear part of the error system dynamics in the nodes. The simulation results suggest existence of a trade off between the number of the channels and the energy of the synchronizer signal, though. In all cases, the conditions for synchronization are expressed in terms of the feasibility of an LMI whose dimension is independent of the number of nodes in the networks. Such LMI's are simple to solve using standard software and the solutions can be used explicitly to design inter-network connections that guarantee synchronization.

5.2 Topics for future research

In this section some ideas for further extension on the results obtained in Chapters 3 and 4 are proposed.

5.2.1 Synchronization without interconnection of all nodes

All through this work we have adhered to a scheme for outer synchronization in which *every* node in one network is connected to its pair in the other network. This assumption may be too restrictive to model some practical systems. A natural extension of the present work, therefore, would involve the modeling and analysis of outer synchronization between networks which are inter-connected through a small subset of nodes only. From this point of view, a problem of immediate interest is the analysis of synchronization of two networks when (a) one of them has attained a state of inner synchronization and (b) the inter-network coupling is carried out through only one node.

5.2.2 Delayed inter-network links

In a large-scale system like a complex network, practical constraints in the links between nodes will often lead to time-delays. In particular, a delay in the information flow from the master network to the slave network may be mathematically reflected by modifying Eq. (4.2) as

$$\begin{aligned} \dot{y}_i(t) &= Ay_i(t) + f\left(y_i(t)\right) + \sum_{j=1}^{N} c_{ij} L y_j(t) + u_i(t), \\ u_i(t) &= \mathcal{K}\left(x_i(t-\tau), y_i(t)\right), \end{aligned} \tag{5.1}$$

where τ is a transmission delay, which can be considered fixed or time-varying; known or unknown. Function $\mathcal{K}(\cdot)$ defines the synchronization signal, and it should be designed such that the synchronization error

$$e(t) = y(t) - x(t-\tau)$$

converges towards 0 over time, i.e., $\lim_{t\to\infty} \|e\,(t)\,\| = 0$. In a more general case, the delay τ can be considered as an unknown but bounded time-varying term, hence schemes that are robust to such variations should be designed as well.

5.2.3 Stochastic uncertainties

In this thesis, we have investigated the effect of the model uncertainties that result from introducing unknown and time-varying, but bounded, perturbations to the model parameters. It is also worth the effort to study the effect of stochastic (noise-like) perturbations of the parameters.

Instead of deterministic fixed bounds, we then usually rely on probabilistic properties of the perturbations. In particular, a "noisy" version of the master and slave coupled networks can be defined as

$$\dot{x}_i(t) = Ax_i(t) + f\left(x_i(t)\right) + \sum_{j=1}^{N} c_{ij} L x_j(t), + g_{1i}(x(t), t) d\omega_i(t), \tag{5.2}$$

and

$$\dot{y}_i(t) = Ay_i(t) + f\left(y_i(t)\right) + \sum_{j=1}^{N} c_{ij} L y_j(t) + u_i(t), + g_{2i}(y(t), t) d\tilde{\omega}_i(t) \tag{5.3}$$

respectively, for $i = 1, \ldots, N$, where $\omega(t)$ and $\tilde{\omega}(t)$ are independent standard Wiener processes, and $g : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ is a continuous function of $x(t)$ that satisfies $g(0, t) \equiv 0$.

5.2.4 Adaptive synchronization

Sometimes the parameters of the systems are initially unknown, or may gradually change over time. The first case may appear when one has a model for a dynamical system but its parameters cannot be directly "observed" and have to be estimated some how. Slow changes in the parameters of a dynamical system (i.e., impedance changes of a resistor when its temperature increases) is an example of the second case. Adaptive control is the method used by a controller which adapts to a system with time-varying parameters which vary, or parameter parameters which are a priori unknown. Adaptive control is different from robust control in that it does not require a priori information about the bounds on these uncertain or time-varying parameters; robust control guarantees that if the changes are within given bounds the control law needs not be changed, while adaptive control is concerned with how the control law itself evolves with time. The structure of an adaptive controller for outer synchronization can be defined as follows. The master and slave networks have the form

$$\dot{x}_i = \hat{A}x_i + \hat{f}\,(x_i) + \sum_{j=1}^{N} c_{ij} L x_j, \quad i = 1, \ldots, N, \tag{5.4}$$

and

$$\dot{y}_i = \hat{A}y_i + \hat{f}\,(y_i) + \sum_{j=1}^{N} c_{ij} L y_j + u_i, \tag{5.5}$$

respectively, where $\hat{A}$ and $\hat{f}(\cdot)$ have the same definition as in Chapter 4, except that they depend on a set of m unknown parameters $\{a_1, a_2, \cdots, a_m\}$. Then rather than designing a controller in the view of, e.g., Chapter 4, we need to devise an update rule for the parameters, of the form $\hat{a}_i(t) = \mathcal{I}\,(x_i(t), y_i(t), \hat{a}_i(t))$ together with the synchronization signal

$$u_i(t) = \mathcal{K}\,(x_i(t), y_i(t), \hat{a}_i(t))\,, \quad i = 1, \ldots, m, \tag{5.6}$$

where the $\hat{a}_i(t)$'s are the estimated values for the parameter $a_i(t)$ at time t. The functions and $\mathcal{K}$ and $\mathcal{I}$ have to be designed jointly to ensure that the new system of equations has a fixed point at $x_i(t) = y_i(t)$ as $t \to \infty$.

5.3 Publications

The main results contained in Chapters 3 and 4 of this thesis have been published in the references

- M. M. Asheghan, J. Miguez, M. T. H. Beheshti, , M. S. Tavazoei, Robust outer synchronization between two complex networks with fractional order dynamics, Chaos, 21 (2011) 033121

- M. M. Asheghan, J. Miguez, Robust global synchronization of two complex dynamical networks, Chaos, 23 (2013) 023108

respectively. *Chaos*, published by the American Institute of Physics (AIP) is currently a top ranked journal in applied mathematics (8th out of 247 entries) and in mathematical physics (7th out of 55 entries), according to the Journal Citation Report 2013, with an impact factor of 2.188.

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